programs. As a result of the discussion it was voted that, beginning with the meeting of January, 1927, the section adopt the policy of pooling the railway fares of all members attending, and the determination of the time and place of the next meeting was delegated to the officers. It is expected that the January, 1926, meeting will be held at Kansas City in connection with the meeting of the American Association for the Advancement of Science.

U. G. MITCHELL, *Secretary-Treasurer.*

IS THE UNIVERSE FINITE?[1]

By ARCHIBALD HENDERSON, University of North Carolina.

1. On one occasion, after finishing a highly theoretical piece of research, the English mathematician H. J. S. Smith is reported to have made the delightful remark:[2] "Thank God, there is something which can never by any possibility have a practical application!" It was perhaps with some such feeling as this that Bolyai and Lobatchewsky developed the non-Euclidean geometry associated with their names, although the latter did dabble a little in parallaxes; and perhaps too, little thought of practical considerations animated the minds of Klein and Newcomb in the analysis of elliptical and spherical space. Little did Ricci and Levi-Civita, I daresay, as they were evolving it, dream of the extraordinary rôle in a new mechanics their theory of the absolute differential calculus was so soon destined to play.

Prophets and seers of the Einsteinian physics and cosmogony, to be sure, were Gauss, Riemann, Helmholtz and Clifford. Despite his dread of the "outcry of the Bœotians," Gauss triangulated in Hanover, using the peaks Inselberg, Brocken and Hoher Hagen, in the effort to discover a deviation of the sum of the angles of a large triangle from two right angles, and thus to decide between Euclidean and non-Euclidean geometry, in regard to validity in space. A herald of today was Clifford who clearly anticipated and foreshadowed the conceptions of Fitzgerald, Lorentz, and Einstein, notably in such a passage as this, from his *Common Sense of the Exact Sciences*:

> "Because a solid figure appears to us to retain the same shape when it is moved about in that portion of space with which we are acquainted, it does not follow that the figure really does retain its shape. The change of shape may be either imperceptible for those distances through which we are able to move the figure, or if they do take place we may attribute them to physical causes,—to heat, light, or magnetism (he should have added 'electricity')—which may possibly be mere names for variations in the curvature of our space."

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[1] This paper, on behalf of the Mathematical Association of America, was delivered on January 1, 1925, before the American Mathematical Society, the Mathematical Association of America, and Sections A, B, and D of the American Association for the Advancement of Science, at Washington, D. C.

[2] Smith's words, probably spoken in regard to some method in the theory of numbers, are thus quoted by D. E. Smith, *History of Mathematics*, I, 467: "It is the peculiar beauty of this method, gentlemen, and one which endears it to the really scientific mind, that under no circumstances can it be of the smallest possible utility."

In his *The Origin and Meaning of the Geometrical Axioms* and cognate papers, Helmholtz gives lucid exposition of fundamental aspects of geometry and space conceptions; and I cannot forbear quoting his words:

"We have no other mark of rigidity of bodies or figures but congruence, whenever they are applied to one another at any time or place, and after any revolution. We cannot however decide by pure geometry and without mechanical considerations whether the coinciding bodies may not both have varied in the same sense."

The foundation stone of relativity is, from the larger aspect, Riemann's study of the differential quadratic form with its creative notion of the "interval," and his extension, to space of any number of dimensions, of Gauss' idea of the curvature of a surface. In his famous memoir: *Concerning the hypotheses which lie at the base of geometry*, delivered at Göttingen in 1854, Riemann uses these memorable words:

"That space is an unbounded three-fold manifoldness is an assumption which is developed by every conception of the outer world; according to which every instant the region of real perception is completed and the possible positions of a sought body are constructed, and which by these applications is forever confirming itself. The unboundedness of space possesses in this way a greater empirical certainty than any external experience; but its infinite extent by no means follows from this; on the other hand, if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite, provided this curvature has ever so small a positive value."

2. Certain physicists and astronomers during the past thirty years have raised the query as to whether the Newtonian mechanics, while holding with remarkable accuracy for our planetary system, also holds exactly for bodies at immeasurably great distances apart. In his paper, "Concerning Newton's law of gravitation" (*Astronomische Nachrichten*, 137, 1895), Seeliger affirms that we must make one or the other assumption: (1) If the common mass of the universe is immeasurably large, then the Newtonian law is not valid as a mathematically strict expression for the controlling powers; or (2) If the Newtonian law is absolutely exact, then the common mass of the universe must be finite. Moreover, either an absolutely empty space or one filled with infinitely tenuous matter would not be in conformity with Newtonian mechanics. For instance, if the universe were infinite and there were an attenuated swarm of fixed stars of approximately the same kind and density, however far we might penetrate the interstellar spaces, then matter would have a finite mean density; and in accordance with Newton's law of gravitation and a theorem due to Gauss, a body at the surface of a very large spherical portion of the universe would be attracted by a force proportional to the product of the radius of the sphere and the mean density of matter. As the radius of the sphere increases without limit, the intensity of the gravitational field at the boundary of the universe would be infinite. This is manifestly impossible, as it would give rise to velocities of a magnitude unobserved by astronomers.

On the other hand, if the mean density of matter were infinitesimally small, the cosmos must present the picture of an island of finite extent surrounded on all sides by infinite empty space. Such a view is repugnant to our minds, since the light of the stars and isolated stars themselves would drift away into the

infinite space devoid of matter, and this ephemeral cosmos would gradually melt away and disappear. Astronomical observation and physical research on the whole do not support the view that the energy of the cosmos is continually being dissipated.[1]

Seeliger advanced two hypotheses to meet the dilemma presented by Newtonian mechanics: either the possibility of matter of negative density in order to produce a null mean density; or to substitute in the numerator of Newton's law of attraction the quantity $e^{-\lambda r}$, where e is the base of natural logarithms, which would require λ to have the value 0.00000038, in order to account for the discrepancy in the advance per century in the perihelion of Mercury, taken as 40″. Professor Hall suggested a modification of Newton's law by which, for great distances, the force of attraction between two masses diminishes more rapidly than would result from the inverse square law, the increment to the exponent 2 being chosen as 0.00000016, in order to explain Mercury's movement. Unfortunately none of these *ad hoc* hypotheses, which have neither empirical nor theoretical foundation, will serve; for while setting right the outstanding anomaly in the node of Venus, as de Sitter has pointed out,[2] they at the same time introduce greater discrepancies in other elements.

3. Is the universe infinite? If a voyager of the skies travel deep into the inter-stellar spaces, past the great blue helium stars of Orion, past Betelgeuse and Antares, beyond the white variable Cepheids, the gaseous red and yellow giant-stars, the faintest of the super-nebulæ, "lying like silver snails in the garden of the stars" but whirling in fiery spirals in the dim void of remoter space—will he ever reach any limit to the universe? Astronomers are not yet agreed that the amount of matter in the material universe is finite. It is significant that the density of matter falls off quite rapidly the deeper we penetrate into the stellar universe. For example, Hale says that there is probably an actual thinning out of the stars towards the boundary of the stellar universe. However, the problem with which we are here concerned is a very different one, *viz.*, has space a curvature? In advancing this question, we are flying in the face of poets, philosophers, scientists, theologians from time immemorial. In his "The dæmon of the world," Shelley says:

> Below lay stretched the boundless universe!
> There, far as the remotest line
> That limits swift imagination's flight,
> Unending orbs mingled in mazy motion,
> Immutably fulfilling
> Eternal Nature's law.
> Above, below, around,
> The circling systems formed
> A wilderness of harmony—
> Each with undeviating aim
> In eloquent silence through the depths of space
> Pursued its wondrous way.

[1] Professor F. R. Moulton regards Seeliger's reasoning as imperfect; and in his *Astronomy* long ago gave an elaborate argument to show that the notion of an infinitely extended universe is not incompatible with a finite mean density.

[2] "On Einstein's theory of gravitation and its astronomical consequences," *M. N. R. A. S.* vol. LXXVI, no. 9, supplementary number, 1916.

On metaphysical grounds alone Kant affirmed that space is infinite and sown with similar stars in all parts. Descartes, confronted with the question, "What lies beyond?" always maintained that a finite universe was impossible. In his *Our Place among Infinities*, the astronomer, Proctor, says: "The teachings of science bring us into the presence of the unquestionable infinities of time and of space, and the presumable infinities of matter and of operation—hence therefore into the presence of infinity of energy." An awful image of an infinite void is procured us by the Book of Job: "He stretcheth out the north over the empty space, and hangeth the earth upon nothing."

In a paper entitled "Cosmological observations concerning general relativity," published in the *Report of the Berlin Academy of Sciences*, February 8, 1917, Einstein advanced the view that the universe is finite, but unbounded. Various and cogent reasons led him to this conclusion. He was in search of a theory of the universe in conformity with the principle of general relativity, by which, in contradistinction to Newtonian mechanics, no preference is given to any reference system, and the laws of nature remain unchanged irrespective of the frame of reference to which they may be assigned. Ignoring the local concentrations of matter, represented by bodies and systems of bodies, Einstein assumed that the matter of the universe is distributed with uniform density—expecting thereby to arrive at some approximate conception of the metrical character of space as a whole. In order to differentiate between the coördinates of space and the coordinate of time, he made the reasonable assumption that the stellar system is approximately at rest, since the motion of matter is very small as compared with the velocity of light. Einstein was now confronted with two alternatives: either to assume that the universe is infinite and Euclidean at infinity; or else to adopt the view of Mach that inertia depends upon a mutual action of matter. Although the former is consonant with our conventional view that Galilean behavior tends to set in as we recede from a massive body, it was rejected by Einstein on the ground that it involves the far-reaching limitation, lacking in a physical basis, namely that B_{iklm} shall vanish at infinity, twenty independent conditions, while only ten curvature components G_{ik} enter into the laws of the gravitational field. It is not difficult, however, to show that, in the important case of the radially symmetrical field, these remaining ten conditions, on being complied with, lead to a solution indistinguishable from the familiar Schwarzschild form—which when r becomes infinite gives the Galilean values for the metrical tensor g_{ik}.

In his choice, Einstein was controlled by his desire to retain Mach's principle of the "relativity of inertia"—according to which the inertia of a body is entirely due to all the remaining matter in the universe. Brief consideration suffices to show that the only set of boundary values at infinity for the g_{ik} which would be invariant for all transformations is that all g_{ik} be zero. This postulate has been termed by de Sitter the "mathematical postulate of relativity of inertia." In Einstein's original theory of 1915, the g_{ik} are determined by the covariant field equations inside matter

$$G_{ik} = -\kappa T_{ik} + \tfrac{1}{2}\kappa g_{ik} T,$$

or

$$G_{ik} - \tfrac{1}{2}g_{ik}G = -\kappa T_{ik},$$
$$G = \kappa T,$$

where T_{ik} is the energy tensor of matter, T is the invariant of the energy tensor, and [1] $g^{ik}T_{ik} = T$.

In the modification of the Newtonian potential used by Neumann, which amounts to replacing the Laplace-Poisson equation by

$$\nabla^2\Omega - \lambda\Omega = -4\pi K\rho,$$

it had been shown by de Sitter that thereby was secured a distribution of matter of a non-vanishing though very small mean density, maintaining its equilibrium without an extra pressure. Guided by this alteration, Einstein modified his original field equations to the form

$$G_{ik} - \lambda g_{ik} = -\kappa(T_{ik} - \tfrac{1}{2}g_{ik}T),$$

which may be written

$$G_{ik} - \tfrac{1}{2}g_{ik}(G - 2\lambda) = -\kappa T_{ik},$$

since

$$G - 4\lambda = \kappa T.$$

On the original assumption that all matter is at rest, with an absence of all pressures, then the tensor T_{ik} becomes

$$T_{44} = g_{44}\rho_0 \text{ and all other } T_{ik} = 0,$$

where ρ_0 is the average density of world matter. Hence the final field equations are:

$$G_{ij} - (\lambda + \tfrac{1}{2}\kappa\rho_0)g_{ij} = 0,$$
$$G_{44} - (\lambda + \tfrac{1}{2}\kappa\rho_0)g_{44} = 0.$$

These can be satisfied by the following:

(*A*) $$ds^2 = -dr^2 - R^2\sin^2\frac{r}{R}(d\psi^2 + \sin^2\psi d\theta^2) + c^2dt^2,$$

if $\kappa\rho_0 = 2\lambda$, $\lambda = 1/R^2$. (Einstein.)

(*B*) $$ds^2 = -dr^2 - R^2\sin^2\frac{r}{R}(d\psi^2 + \sin^2\psi d\theta^2) + \cos^2\frac{r}{R}c^2dt^2,$$

if $\rho_0 = 0$, $\lambda = 3/R^2$. (de Sitter.)

(*C*) $$ds^2 = -dr^2 - r^2(d\psi^2 + \sin^2\psi d\theta^2) + c^2dt^2,$$

if $\rho_0 = 0$, $\lambda = 0$. (Newton.)

[1] Here $\kappa = 8\pi K/c^2$ where $K = (6{\cdot}6)10^{-8}$ and $c = (3)10^{10}$ in C.G.S. units.

The transformation $r = R\chi$ immediately gives for the *three-dimensional line-element* in both (A) and (B)

$$d\sigma^2 = R^2\{d\chi^2 + \sin^2\chi(d\psi^2 + \sin^2\psi d\theta^2)\}$$

—easily recognizable as representing *Riemann spherical space*, with constant curvature, $1/R^2$. For very great distances, it is easy to show that (B) but not (A) satisfies the "mathematical postulate of relativity." In this respect, de Sitter's space-time world is preferable to Einstein's. Einstein escaped the problem of boundary conditions at infinity by considering the world to be cylindrical—curved in the three space dimensions and straight in the time dimension. This required him to postulate the existence of vast quantities of matter, called by de Sitter "world-matter," far in excess of the amount now known to exist. It is easy to show that for Einstein's space-time world

$$2/R^2 = \kappa\rho_0 = 8\pi K\rho_0/c^2,$$

whence

$$R = \sqrt{2/\kappa\rho_0}, \qquad M = \rho_0 \cdot 2\pi^2R^3 = 4\pi^2R/\kappa = \sqrt{32}\,\pi^2/\sqrt{\rho_0\kappa^3},$$

where

$$\kappa = (1\cdot87)10^{-27}$$

and

$$2/\kappa = (1\cdot08)10^{27}.$$

It is obvious that if $\rho_0 = 0$, $R = \infty$—*i.e.*, an infinite world.

4. I shall not regale you here with a description of the fantastic properties of the Einstein and de Sitter space-time universes, each finite but unbounded in their space extensions. Of the two types of elliptical space, Einstein deals with the antipodal, de Sitter with the polar. In the case of the former

$$M = 7\cdot10^{41}/\sqrt{\rho_0}, \qquad V = 7\cdot10^{41}/\sqrt[4]{\rho_0}$$

of the latter

$$M = \rho_0 \cdot \pi^2R^3, \quad \text{etc.}$$

The formula for the Einstein world, $2/R^2 = \kappa\rho_0$, connecting R and ρ_0 may be described as "arresting" and "captivating," since, in the present state of astronomical knowledge, neither the density nor the world-radius can be arrived at with any degree of accuracy. It is self-protecting, since as we penetrate farther into space picking up very remote celestial objects, we reduce enormously the average density and thereby increase correspondingly the predicted radius. Were the spiral nebulæ, for example, found to be at distances greater than a computed world-radius, a new value of the world-radius greater than the former value would arise from the new value of the mean density. All that can be done, from this mode of approach, is to set up reasonable assumptions, and see if the conclusions arrived at are consistent. For example, the latest estimates of

the diameter of our galaxy range from 30,000 (Curtis) to 300,000 light-years (Shapley)—the yardstick here being the distance that light, with a velocity of 186,000 miles per second, travels in a year. Hence the radius of our spherical universe, if there are no extra-galactic objects, must be at least 150,000 light-years. The average mean density in the universe may be taken as equal to that of the Milky Way. Using as a yardstick the distance from the earth to the sun, namely 93 million miles, the radius of the Einstein universe is one million times ten millions times the distance from the earth to the sun. It would take a ray of light one billion years to go around the Einstein universe. The weight of this universe, in grams, would be 10^{54}—the weight of a trillion suns.[1]

In his paper, "On Einstein's theory of gravitation," [2] de Sitter has made certain interesting computations for the radius of the finite universe of the polar type, so designed as not to contradict the known data of astronomical observations; and these are given in astronomical units. On the assumption that some of the spiral nebulæ or globular clusters are galactic systems comparable with our own in size, he finds from the formula for apparent angular diameter for elliptical space, at the greatest distance $\frac{1}{2}\pi R$, that

$$R \geqq 10^{12}.$$

If we choose for the mean density of the universe the star-density at the center of the galactic system, *viz.*, $\rho_0 = 10^{-17}$, we find

$$R = (9)10^{11}.$$

It is very probable that in the part of space which immediately surrounds our galactic system there are many similar systems whose mutual distances are large compared with their dimensions. If we assume as the average shortest distance between neighboring systems 10^{10}; and further suppose that the whole universe is thus filled with galactic systems, each of mass $\frac{1}{3}10^{10}$, then $\rho_0 = \frac{1}{3}10^{-20}$ and

$$R \leqq (5)10^{13}.$$

From the consideration of certain selective attenuation effects on starlight as due to molecular scattering, King finds that the density of interstellar residual gas amounts to 6,300 suns per cubic parsec. Using Shapley's determination, which is one fiftieth of this, we find

$$\rho_0 = (1\cdot5)10^{-14},$$

giving

$$R = (2)10^{10}.$$

In de Sitter's universe, $g_{44} = \cos^2 \chi$; and hence the frequency of light-vibration, derived according to the familiar method, diminishes with increasing

[1] In this connection consult the writer's paper, "The size of the universe," *Science*, September 7, 1923.

[2] *M. N. R. A. S.*, vol. 78, 1917, 8 (no. 1, 1917).

distance from the origin of coördinates. Consequently on the assumption of the permanence of atoms, we may expect the spectrum lines of very distant stars or nebulæ to be displaced systematically toward the red. Assigning one third of this systematic displacement as an Einstein-effect due to the star's own gravitational field, the rest—corresponding to a receding radial velocity of 3 km./sec. for the helium or B stars—may be set down as an apparent displacement due to the diminution of g_{44}. Since the average distance of the helium or B stars has been computed to be about $(3)10^7$, we find

$$R = (0{\cdot}67)10^{10}.$$

Applying like reasoning to the Lesser Magellanic Cloud, which shows a spectrum shift corresponding to a radial velocity of + 150 km./sec., with $r > (6)10^9$ (Hertzsprung), de Sitter finds

$$R > (2)10^{11}.$$

It is worthy of note that Weyl's new theory (1918), welding together electricity and gravitation, suggests a new mode of computing the curvature radius of space-time. The theory indicates that the ratio of the gravitational to the electrical radius of an electron should be of the same order as the ratio of the electrical radius of an electron to the radius of curvature of the world. Now the gravitational mass or "radius" of an electron is $(7)10^{-56}$ cms., the electrical radius $(2)10^{-13}$ cms. Hence it follows that the radius of space would have to be of the order $(6)10^{29}$ cms. or $(2)10^{11}$ parsecs. Remembering that 1 parsec equals $(2{\cdot}06)10^5$ astronomical units, this gives a value of R appreciably larger than those found by de Sitter.

5. In a series of letters and papers,[1] Silberstein has followed up the suggestion of de Sitter, and utilized results known with reasonable accuracy concerning certain remote celestial objects, in the effort to arrive at a closer value for R. For the elliptical space of de Sitter he has derived a formula for the Doppler shift, $d\lambda/\lambda$, of a star in inertial relative motion to the observer

$$d\lambda/\lambda = \left(1 - \frac{v_0^2}{c^2}\right)^{-1}\left\{1 \pm \left[1 - \cos^2\frac{r}{R}\Big/\left(1 - \frac{v_0^2}{c^2}\right)\right]^{1/2}\right\} - 1,$$

which for small or moderate values of r/R reduces to $d\lambda/\lambda = \pm\, r/R$, where

v_0 = individual velocity of star at some past or future epoch,
c = velocity of light,
r = distance of star from observer,
R = curvature radius of space-time,

since the expression $(1 - (v_0^2/c^2))^{-1/2}$ as suggested by astronomical observations, according to Lundmark, will be ≤ 1.0000444 and for most of the stars ≤ 1.000001. There is excellent reason for regarding the spectral displacements as of the

[1] *Nature*, January 1, March 8, April 26, June 7, September 6, 1924; *M. N. R. A. S.*, March 1924; *Philosophical Magazine*, May and October 1924.

Doppler type; but it is as yet an open question whether or not the spectral shifts are due to real motion and show no influence of the world-curvature. Contrary to the formulas used by Weyl and Eddington, Silberstein's formula has a $\pm$ sign, which accounts as well for receding as for approaching velocities. By using eight globular clusters and the Greater and Lesser Magellanic Clouds, Silberstein finds as a mean value:

$$R = (5{\cdot}9)10^{12}$$

or about *six million million* astronomical units.

Employing a statistical formula, an equation for R in terms of distances and Döppler effects alone,

$$\overline{D}_1^2 - \overline{D}_2^2 = \frac{1}{R^2}(\bar{r}_1^2 - \bar{r}_2^2),$$

when $D = d\lambda/\lambda$, obtained by choosing a mean distance for different groups, Silberstein finds for eleven globular clusters and two clouds:

$$R = (8{\cdot}8)10^{12}.$$

Again, giving up what he regards as the "artificial limitation" to radial motions, Silberstein has derived a spectrum-shift formula for any inertial motion, which when applied to the same thirteen celestial bodies, as before, gives:

$$R = (7{\cdot}2)10^{12}.$$

6. In the latest contribution to this subject,[1] the work of Silberstein is sharply criticized by Lundmark. Globular clusters are not regarded as nearly far enough away to be appreciably affected by the slowing down of atomic vibrations; and even if used, Silberstein is taken to task for not using all sixteen known velocities—instead of only those which give a rather constant value of the R. Silberstein rejected the globular clusters having a small radial velocity, *i.e.*, markedly under 100 km./sec. Using all the available data (eighteen in all, including two unpublished, supplied by Dr. Slipher), Lundmark found the values of R to range from $(2{\cdot}3)10^{12}$ to $(96{\cdot}5)10^{12}$ (Shapley's Parallaxes) and from $(7{\cdot}4)10^{12}$ to $(62{\cdot}5)10^{12}$ (Lundmark's Parallaxes)—a dispersion in R so large as to warrant the prediction that, for the present at least, the curvature of space-time cannot be determined with any accuracy by using the displacements in the spectra of globular clusters.

Lundmark goes on to examine other clusters of rather distant objects: Cepheids, Novæ, the O stars, the eclipsing variables, certain classes of red stars, and the spiral nebulæ. Nothing favorable to Silberstein's value of between 6 and 7 times 10^{12} astronomical units for R arises until we come to the O stars, which, on using the mean distance 1,560 parsecs, give the value

$$R = (4)10^{12},$$

[1] *M. N. R. A. S.*, October, 1924.

but unfortunately the values computed from individual velocities show a very large dispersion. The eclipsing variables (31 velocities) give the value (using mean distance):

$$R = (2\cdot7)10^{12}.$$

The R stars (29) give values for R ranging from $(0.6)10^{12}$ to $(6\cdot7)10^{12}$. The N stars (192) give, using mean radial velocity 18.8 km./sec.,

$$R = (2\cdot3)10^{12}.$$

The real interest centers in the results given by the spiral nebulæ, which from the figures of Curtis (66 objects), Lundmark (82), Van Maanen (7 objects) and the same (Smart's values) give the following table:

R	*Material*
$(1\cdot0)10^{12}$	Curtis
$(0\cdot9)10^{12}$	Lundmark
$(6\cdot6)10^{12}$	Van Maanen
$(5\cdot2)10^{12}$	Van Maanen (Smart's values).

The last two results are very striking, as tending to confirm Silberstein. But Lundmark warns against their too ready acceptance, because the distances are probably too small; and further, because the parallaxes were derived by using the Döppler effects as indicating true motions, whereas the above procedure involved the supposition that the spectral shifts represent radial motions only.

The five values of R found by de Sitter ranging from

$$(0\cdot67)10^{10} \text{ to } (5)10^{13},$$

which showed an agreement not to be expected *à priori;* the values of Silberstein ranging from

$$(5\cdot9)10^{12} \text{ to } (8\cdot8)10^{12}$$

and the cited values of Lundmark, *viz.*,

$$(2\cdot7)10^{12},\ (4)10^{12},\ \text{and}\ (1\cdot0)10^{12} \text{ to } (6\cdot6)10^{12},$$

indicate a tendency to accumulate in the neighborhood of

$$(6)10^{12} \text{ to } (8)10^{12}.$$

Lundmark is in agreement with Silberstein that the latter's formula, in which the v_0^2 for one group of objects is assumed to be the same for another group placed at another distance, is practicable for application to moving clusters and some of the open clusters.

Recent investigations, reported by H. D. Curtis,[1] indicate with something like certainty that the spiral nebulæ are isolated stellar systems, at least a hundred

[1] Meeting of the A. A. A. S., in a paper read at the Bureau of Standards, December, 1924.

million light-years away.[1] Shapley was the first to point out and insist upon the significance of the systematic recessional motions of the spiral nebulæ. Whether above or below the plane, they recede from the Milky Way at high speed, all except the brighter and nearer ones. The displacement of the spectral lines to the red was thus interpreted as a Doppler effect. If, as now appears probable, the spirals are isolated stellar systems, this recession must be explained, it appears, either as a wholesale error or else as a relativistic effect. The former view is untenable; the latter lends strong support to the notion of a finite universe.[2] Much additional data will be required and many further researches made before it will be possible categorically to decide between the infinite, limitless, Euclidean universe of Newton and the finite, unbounded, non-Euclidean universe of Einstein or of de Sitter.

THE EULER DIFFERENTIAL EQUATION OF INFINITE ORDER.

By H. T. DAVIS, Indiana University.

1. **The Problem.** Originating in the important memoirs of C. Bourlet and S. Pincherle published in 1897,[3] the method of solving functional equations by first reducing them to differential equations of infinite order seems to promise a fruitful field for mathematical investigation. Although several articles on the subject have recently appeared in foreign journals, the theory does not seem to be generally known nor widely used. It is with the idea of indicating the application of the methods of this new calculus to the problems of analysis that the author in the present paper has developed the theory of an elementary type of differential equation of infinite order. In the last section of the paper an application, thought to be new, is made of the so-called "reduction principle" (principe des réduites) of the theory of linear equations in an infinite number of unknowns. The use of this principle in the present problem is of especial interest since it affords in many cases a method for computing the zeros of a function defined by a factorial series.

In this paper we propose to discuss the following Euler differential equation of infinite order:

$$F(x) = a_0\phi(x) + a_1 x\phi'(x) + \frac{a_2}{2!}x^2\phi''(x) + \frac{a_3}{3!}x^3\phi'''(x) + \cdots, \qquad (1)$$

where the a_n are bounded as $n \to \infty$ and $F(x)$ is analytic about $x = 0$.

We may observe that we are led to an equation of this type in solving the

[1] This is one hundred times the figure suggested by Eddington in his *Space, Time and Gravitation* (1921).

[2] Letter of Professor Harlow Shapley to the writer, November 16, 1924.

[3] C. Bourlet: *Annales de l'École Normal Supérieure*, 3d series, vol. 14 (1897), pp. 133–190; S. Pincherle: *Mathematische Annalen*, vol. 49 (1897), pp. 325–381.

following rather general type of functional equation:

$$\int_0^x \frac{\phi(t)}{(x-t)^\nu}dt = x^{1-\nu}\sum_{i=0}^{m} c_i\phi(\mu_i x) + f(x), \qquad 0 \leqq \nu < 1, \tag{2}$$

where $f(x)$ is a known function of the form $x^{1-\nu}g(x)$, $g(x)$ being analytic about $x = 0$, and where c_i and μ_i are any set of constants.[1]

It will be seen that this equation generalizes the functional equation,

$$\int_0^x \phi(t)dt = \frac{x}{6}[\phi(0) + 4\phi(\tfrac{1}{2}x) + \phi(x)], \tag{3}$$

which has attracted considerable attention in the MONTHLY.[2] It also includes as special cases Abel's well-known integral equation,

$$\int_0^x \frac{\phi(t)}{(x-t)^\nu}dt = f(x),$$

and Weddle's formula for which a solution has recently been sought.[3]

We now assume that $\phi(t)$ and $\phi(\mu_i x)$ can be developed into the following Taylor's series:

$$\phi(t) = \phi(x) + (t-x)\phi'(x) + \frac{(t-x)^2}{2!}\phi''(x) + \cdots,$$

$$\phi(\mu_i x) = \phi(x) + (\mu_i - 1)x\phi'(x) + \frac{(\mu_i - 1)^2}{2!}x^2\phi''(x) + \cdots.$$

When these values have been substituted in equation (2), we obtain a differential equation of Euler type where $F(x) = x^{\nu-1}f(x)$ and

$$a_n = (-1)^n\left[\frac{1}{n-\nu+1} - \sum_{i=0}^{m} c_i(1-\mu_i)^n\right].$$

In equation (3) these values of a_n are easily seen to reduce to

$$(-1)^n\left[\frac{1}{n+1} - \frac{1}{6} - \frac{1}{3\cdot 2^{n-1}}\right]$$

and for Abel's equation we have

$$a_n = \frac{(-1)^n}{n+1-\nu}.$$

[1] This equation is related to a functional equation studied by P-J. Browne: *Annales de Toulouse*, vol. 4 (3), pp. 63–198. See also C. Popovici: *Comptes Rendus*, vol. 158, pp. 1866–1869.

[2] MONTHLY: Question **34** (*1917*, 134, 341; *1920*, 114, 301, 405, 460; *1921*, 19).

[3] MONTHLY: Problem **3076** (*1924*, 254).

2. The Non-Homogeneous Equation. Because of our assumption as to the analyticity of $F(x)$ we may differentiate [1] equation (1) n times and thus obtain the following system of linear equations in the infinitely many unknowns ϕ, ϕ', ϕ'', $\cdots$:

$$\begin{aligned}
F(x) &= a_0\phi + a_1 x\phi' + \frac{a_2}{2!}x^2\phi'' + \cdots + \frac{a_n}{n!}x^n\phi^{(n)} + \cdots,\\
F'(x) &= (a_0+a_1)\phi' + (a_1+a_2)x\phi'' + \cdots + \frac{a_{n-1}+a_n}{(n-1)!}x^{n-1}\phi^{(n)} + \cdots,\\
F''(x) &= (a_0+2a_1+a_2)\phi'' + \cdots + \frac{a_{n-2}+2a_{n-1}+a_n}{(n-2)!}x^{n-2}\phi^{(n)} + \cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
F^{(n)}(x) &= (a_0+na_1+\cdots+a_n)\phi^{(n)} + \cdots.
\end{aligned} \tag{4}$$

Suppressing terms which contain derivatives of order higher than n, we are left with a system of $n+1$ linear equations in the $n+1$ unknowns ϕ, ϕ', $\cdots$, $\phi^{(n)}$, which we can solve formally for ϕ. If we designate by D_n the determinant of the system and by D_{0n}, D_{1n}, $\cdots$, D_{nn} the cofactors of the elements of the first column, then this formal solution, which we may denote by ϕ_n, will be

$$\phi_n = \frac{F(x)D_{0n} + F'(x)D_{1n} + \cdots + F^{(n)}(x)D_{nn}}{D_n}. \tag{5}$$

When the limit of the right-hand member exists for $n \to \infty$, we shall call $\lim_{n=\infty} \phi_n = \phi(x)$ the solution of the differential equation of infinite order if, for this function, the right-hand member of the equation converges uniformly to $F(x)$.

Denoting by $D(n)$ the product

$$D(n) = a_0(a_0 + a_1)(a_0 + 2a_1 + a_2)\cdots\left(a_0 + na_1 + \frac{n(n-1)}{2!}a_2 + \cdots + a_n\right),$$

and assuming that $D(n) \neq 0$, the formal solution (5) is easily seen to reduce to

$$\begin{aligned}
\phi_n(x) = \frac{1}{D(0)}F(x) - \frac{\Delta_1}{D(1)}xF'(x) + \frac{\Delta_2}{D(2)}x^2F''(x) - \cdots\\
+ (-1)^n\frac{\Delta_n}{D(n)}x^nF^{(n)}(x),
\end{aligned} \tag{6}$$

where Δ_1, Δ_2, $\cdots$, Δ_n are the leading principal minors of the first, second, $\cdots$, nth orders respectively of the determinant formed from the constant coefficients of the cofactor D_{nn}. It is obvious that this solution does not exist if any of the factors of $D(n)$ are equal to zero; the discussion of this case will be postponed to §4.

[1] For a treatment of the general problem by this method see E. Hilb: *Mathematische Annalen*, vol. 82 (1920–21), pp. 1–39.

Formula (6) will be recognized as the analogue of the operator

$$\left[\frac{1}{a_0 + a_1 D_x + a_2 D_x^2 + \cdots + a_n D_x^n + \cdots}\right] F(x),$$

which has long been known to give the formal solution of the non-homogeneous differential equation with constant coefficients.[1]

It is clear that series (6) will converge provided

$$\lim_{n=\infty} \sqrt[n]{\frac{\Delta_n}{D(n)} F^{(n)}(x)} = 0,$$

but it is desirable to find a criterion which applies separately to $F^{(n)}(x)$ and $\Delta_n/D(n)$.

If we make use of Hadamard's well-known theorem on the maximum value of a determinant, which states that the absolute value of a determinant does not exceed the square root of the product of the sums of the squares of the absolute values of the elements of each row,[2] we arrive at the inequality,

$$\left|\frac{\Delta_n}{D(n)}\right| \leqq \frac{\sqrt{\sigma_0 \sigma_1 \cdots \sigma_{n-1}}}{|a_0 + na_1 + \cdots + a_n|},$$

where for brevity we have set

$$\sigma_r = 1 + \sum_{m=1}^{n-r} \left(\frac{a_m + ra_{m+1} + \dfrac{r(r-1)}{2!} a_{m+2} + \cdots}{a_0 + ra_1 + \dfrac{r(r-1)}{2!} a_2 + \cdots} \right)^2 \left(\frac{1}{m!}\right)^2.$$

Then if constants A and K can be found such that we have

$$\left(\frac{a_n + ra_{n+1} + \cdots}{a_0 + ra_1 + \cdots}\right)^2 \leqq n!A \quad \text{and} \quad |a_0 + na_1 + \cdots + a_n| > K \geqq 1, \quad (7)$$

for all values of r and n, we shall have

$$\lim_{n=\infty} \sqrt[n]{\left|\frac{\Delta_n}{D(n)}\right|} < (1 + Ae)^{1/2} = C.$$

We thus arrive at the result that, *if the coefficients of the differential equation satisfy conditions* (7), *and* $D(n) \neq 0$ *for all values of* n, *series* (6) *will converge provided* $\lim_{n=\infty} \sqrt[n]{F^{(n)}(x)} = 0$.

[1] For a complete discussion of this operator see F. Schürer: *Leipziger Berichte*, vol. 70 (1918), pp. 185–246.

[2] J. Hadamard: Résolution d'une question relatif aux determinants. *Bull. des. Sc. Math.*, vol. 17 (1893), pp. 240–246.

We must next show that when series (6) is substituted in the differential equation, the right-hand member converges uniformly to $F(x)$. In order to do this, let us denote by S_n the series

$$S_n = a_0\phi(x) + a_1x\phi'(x) + \frac{a_2}{2!}x^2\phi''(x) + \cdots + \frac{a_n}{n!}x^n\phi^{(n)}(x),$$

and for simplicity write

$$\phi(x) = A_0F(x) + A_1xF'(x) + \frac{A_2}{2!}x^2F''(x) + \cdots + \frac{A_n}{n!}x^nF^{(n)}(x) + \cdots,$$

where $A_n = (-1)^n n!(\Delta_n/D(n))$.

We wish to show that the difference $A(x) = F(x) - S_n(x)$ approaches zero uniformly as $n \to \infty$.

Since we have

$$a_0\frac{A_r}{r!} + a_1\frac{A_{r-1} + A_r}{(r-1)!} + \frac{a_2}{2!}\frac{(A_{r-2} + 2A_{r-1} + A_r)}{(r-2)!} + \cdots + \frac{a_r}{r!}(A_0 + rA_1 + \cdots + A_r) \equiv 0$$

for all values of r, it follows that

$$A(x) = \left\{a_0A_{n+1} + a_1(n+1)(A_n + A_{n+1}) + \frac{a_2}{2!}(n+1)n(A_{n-1} + 2A_n + A_{n+1}) + \cdots + \frac{a_n}{n!}(n+1)!\,(A_1 + nA_2 + \cdots + A_{n+1})\right\}\frac{x^{n+1}}{(n+1)!}F^{(n+1)}(x) + \cdots.$$

Then since the a_n are bounded, $|a_n| < a$, and since

$$|A_{n+k}| = (n+k)!\left|\frac{\Delta_{n+k}}{D(n+k)}\right| < (n+k)!\,C^{n+k}, \quad n > n',$$

it will be seen that

$$|A(x)| < aC^{n+1}\left[1 + (n+1)2 + \frac{(n+1)n}{2!}2^2 + \cdots + \frac{(n+1)!}{n!}2^n\right]$$
$$|x|^{n+1}|F^{(n+1)}(x)| + \cdots < aC^{n+1}3^{n+1}|x|^{n+1}|F^{(n+1)}| + aC^{n+2}3^{n+2}|x|^{n+2}|F^{(n+2)}| + \cdots.$$

But by hyopthesis on $F(x)$ we can always find an n' to match an arbitrarily small ϵ, so that $|F^{(n)}(x)| < \epsilon^n$, for $n > n'$ and $|x| < k$.

Hence it follows that

$$|A(x)| < a(C3k\epsilon)^{n+1}[1 + 3k\epsilon + (3k\epsilon)^2 + \cdots] = a\frac{(C3k\epsilon)^{n+1}}{1 - 3k\epsilon}.$$

Consequently if ϵ is chosen so that $\epsilon < 1/3Ck$, it is clear that $A(x)$ will approach zero uniformly as $n \to \infty$.

Another form may be given to series (6) by replacing $F^{(m)}(x)$ by the equivalent Cauchy integral

$$F^{(m)}(x) = \frac{m!}{2\pi i}\int_C \frac{F(t)}{(t-x)^{m+1}}\,dt,$$

so that we have

$$\lim_{n=\infty}\phi_n(x) = \frac{1}{2\pi i}\int_C \frac{F(t)}{t-x}\left[\frac{1}{D(0)} - \frac{\Delta_1}{D(1)}\frac{x}{t-x} + \frac{2!\,\Delta_2}{D(2)}\frac{x^2}{(t-x)^2} - \cdots\right]dt.$$

It has been proved by Bourlet that the right-hand member of this equation will exist for all functions $F(t)$, analytic in a region of radius ρ about x, provided the expression in brackets converges for all values of t satisfying the condition

$$|t-x| = \rho.$$

3. The Method of Operators. We shall next apply to the problem the method of operators as developed in Bourlet's memoir already referred to. We can write equation (3) in the symbolic form

$$F(x) = \left(a_0 + a_1 x\frac{d}{dx} + \frac{a_2}{2!}x^2\frac{d^2}{dx^2} + \cdots\right)\phi(x). \tag{8}$$

The function

$$\Theta(xz) = a_0 + a_1 xz + \frac{a_2}{2!}x^2z^2 + \cdots \tag{9}$$

has been called the *generatrix function* and is of fundamental importance in the calculus of operators.[1]

It is easily established by a theorem due to Hadamard [2] that (9) is an entire function in xz of genus 1 or 0.

Therefore $\Theta(xz)$ may be written

$$\Theta(xz) = e^{cxz}\prod_{i=0}^{\infty}\left(1 - \frac{xz}{\beta_i}\right), \tag{10}$$

where the β_i are the zeros of the entire function $\Theta(y)$.

Product operators of this kind have been little studied. In a paper published in 1917 [3] J. F. Ritt has developed the properties of an operator of genus zero of the form

$$A = \prod_{i=0}^{\infty}\left(1 - \frac{D}{\alpha_i}\right)$$

where it is assumed that $\sum_{i=1}^{\infty}\frac{1}{|\alpha_i|}$ converges.

[1] We employ here the name used by T. Lalesco: *Journal de Mathématique*, ser. 6, vol. 4 (1908), p. 193. Bourlet calls this function the "fonction opérative."

[2] J. Hadamard: *Journal de Mathématique*, vol. 9 (4) (1893), p. 172. See also Bourlet: *loc. cit.*, p. 162.

[3] J. F. Ritt: *Trans. of the Amer. Math. Soc.*, vol. 18 (1917), pp. 27–49.

In case the number of zeros is finite, however, the analysis becomes much easier, and we can then write (10) in the form

$$\Theta(xz) = e^{cxz}P(xz),$$

where $P(xz)$ is a polynomial.

Bourlet has proved that the generatrix of the operator $A\left(x, \frac{d}{dx}\right)\left[B\left(x, \frac{d}{dx}\right)\right]$, where A and B are linear operators and the symbol denotes the operation of A on B, can be obtained from the following formula in which $A(x, z)$ and $B(x, z)$ are the generatrix functions of the two operators. Thus we have

$$\text{Generatrix}\left\{A\left(x, \frac{d}{dx}\right)\left[B\left(x, \frac{d}{dx}\right)\right]\right\} = B(x, z)\cdot A(x, z) + \frac{\partial B}{\partial x}\frac{\partial A}{\partial z} + \frac{1}{2!}\frac{\partial^2 B}{\partial x^2}\frac{\partial^2 A}{\partial z^2} + \cdots + \frac{1}{n!}\frac{\partial^n B}{\partial x^n}\frac{\partial^n A}{\partial z^n} + \cdots.$$

Applying this formula to the functions $A(x, z) = e^{cxz}$ and $B(x, z) = B(xz)$, we obtain

$$\text{Generatrix}\left\{A\left(x, \frac{d}{dx}\right)\left[B\left(x, \frac{d}{dx}\right)\right]\right\} = e^{cxz}B[(1 + c)xz].$$

Since this is the generatrix $\Theta(xz)$, we must have

$$P(xz) = B[(1 + c)xz]$$

and we see that the original operator can be resolved into the product of two operators. Hence we may write equation (8) in the form

$$e^{cx(d/dx)}\left\{P\left(\frac{x\frac{d}{dx}}{1 + c}\right)\phi(x)\right\} = F(x).$$

If we operate on both sides of this equation by the inverse of $e^{cx(d/dx)}$, it follows from a simple application of formula (6) that we shall obtain

$$P\left(\frac{x\frac{d}{dx}}{1 + c}\right)\phi(x) = F(x) - \frac{c}{1 + c}xF'(x) + \frac{c^2}{(1 + c)^2}\frac{x^2}{2!}F''(x) - \cdots,$$
$$= F\left(\frac{x}{1 + c}\right).$$

The problem of solving equation (8) is thus reduced to that of solving an Euler equation of finite order.

Let us consider the following example:

$$\tfrac{1}{4}x^2 = \phi(x) + \tfrac{1}{2}x\phi'(x) + \left(\frac{1}{2^2 2!} - 1\right)x^2\phi'' + \left(\frac{1}{2^3 3!} - \tfrac{1}{2}\right)x^3\phi''' + \cdots. \quad (11)$$

The generatrix function is easily seen to be

$$\Theta(xz) = e^{(1/2)xz}(1 - x^2z^2).$$

The equation is then equivalent to

$$e^{(1/2)x(d/dx)}\left(\phi - \frac{4}{9}x^2\frac{d^2\phi}{dx^2}\right) = \tfrac{1}{4}x^2,$$

which reduces to

$$\tfrac{4}{9}x^2\phi'' - \phi = -\tfrac{1}{9}x^2.$$

The solution of this equation is $\phi = c_1x^{\alpha_1} + c_2x^{\alpha_2} + x^2$, where $\alpha_1 = \frac{1}{2}(1 + \sqrt{10})$ and $\alpha_2 = \frac{1}{2}(1 - \sqrt{10})$. By direct substitution, this solution will be found to satisfy the original equation as well.

4. The Homogeneous Equation. In the last section we saw how the method of operators could be used to reduce the problem to an equation of finite order, provided the generatrix had a finite number of zeros. In general this will not be the case so we must approach the homogeneous problem in another way.

By analogy with the method used in solving the Euler equation of finite order, we seek solutions of the form $\phi = x^r$ and are thus led to the auxiliary function

$$f(r) = a_0 + a_1r + \frac{a_2}{2!}r(r-1) + \frac{a_3}{3!}r(r-1)(r-2) + \cdots, \tag{12}$$

which we observe is a factorial series.[1] The region of convergence of this series is, in general, a half plane, limited on the left by a straight line perpendicular to the axis of reals and cutting it in a point λ, called the abscissa of convergence. In particular, the value of λ may be $-\infty$. Within the domain of convergence, then, for every simple zero r_i of this function there will exist one solution of the differential equation of the form

$$\phi(x) = x^{r_i},$$

and for every multiple root r_i of multiplicity μ_i there will exist μ_i solutions. These solutions will be

$$x^{r_i}, \quad x^{r_i}\log x, \quad \cdots, \quad x^{r_i}(\log x)^{\mu_i - 1}.$$

It is thus seen that the solution of the homogeneous equation depends upon the convergence of series (12) and upon the determination of the zeros of the function defined by this series. A simple sufficiency condition for the convergence of (12) is given by the following lemma:

LEMMA. *The abscissa of convergence of* (12) *is never greater than the smallest*

[1] For a study of the properties of this factorial series, see S. Pincherle: Sur les fonctions déterminantes, *Annales de l'École Normale Supérieure*, vol. 22 (3), (1905), pp. 57–68.

value of p for which the series

$$\sum_{n=1}^{\infty} \frac{|a_{n+1}|}{n^{p+1}}, \tag{13}$$

converges.

To prove this let

$$P_n(r) = \left| \frac{a_{n+1}}{(n+1)!} r(r-1) \cdots (r-n) \right| = \left| \frac{a_{n+1} \prod_{k=1}^{n} r\left(1 - \frac{r}{k}\right) e^{r/k}}{(n+1) \prod_{k=1}^{n} e^{r/k}} \right|.$$

Since $\prod_{k=1}^{n} r\left(1 - \frac{r}{k}\right) e^{r/k}$ is a convergent product, there will exist a number N such that for all values of $n > N$ the absolute value of the product will never exceed a value K.

Now replace r by $p + qi$. Then we shall have

$$P_n(r) \leqq \left| \frac{a_{n+1}K}{(n+1) \prod_{k=1}^{n} e^{p/k} e^{(q/k)i}} \right| \leqq \left| \frac{a_{n+1}K}{(n+1) \prod_{k=1}^{n} e^{p/k}} \right|, \quad n > N.$$

But since $\lim (1 + \frac{1}{2} + \cdots + (1/n) - \log n) = C$, where C is Euler's constant, there will exist a number N' such that for $n > N'$ we shall have

$$1 + \tfrac{1}{2} + \cdots + \frac{1}{n} > C + \log n - \epsilon,$$

where ϵ is an arbitrarily chosen positive number independent of n.

Then, for n greater than N and N', we have

$$P_n(r) \leqq \frac{K}{e^{p(C-\epsilon)}} \frac{|a_{n+1}|}{(n+1)n^p} < \frac{K}{e^{p(C-\epsilon)}} \frac{|a_{n+1}|}{n^{p+1}}.$$

If $\bar{p}$ is the smallest value of p for which the series (13) converges, then it will clearly converge for all greater values of p. Hence the abscissa of convergence is equal to or less than $\bar{p}$ and series (12) is uniformly convergent in the plane to the right of $\bar{p}$.

Convergence theorems for series (13) are well known since it is a Dirichlet's series which has been extensively studied.[1]

We now consider an interesting special case, already referred to in section 2, where one or more of the factors of $D(n)$ are zero. It is easily seen that if the $(m-1)$st factor of this product is zero, then the factorial series (12) will vanish for $r = m$.

[1] G. H. Hardy and M. Riesz: *The General Theory of Dirichlet's Series.* Cambridge, 1915.

Thus in (12) let $r = m$ and we shall have

$$f(m) = \left[a_0 + ma_1 + \frac{m(m-1)}{2!} a_2 + \cdots + a_m \right] + \frac{a_{m+1}}{(m+1)!} m(m-1) \cdots (m-m) + \cdots,$$

which clearly reduces to zero.

In order to obtain the solution of the non-homogeneous differential equation corresponding to this case, we may let $\phi = x^m u(x)$ and thus obtain a new equation of the form

$$x^{-m-1} F(x) = A_1 u'(x) + A_2 x u''(x) + \cdots. \tag{14}$$

The left-hand member of this equation will be analytic about $x = 0$ provided $F(x)$ has a zero of order $m + 1$ at the origin and we can then solve for $u'(x)$ by the method given in section 2. By repetition of this process all of the zero factors of $D(n)$ can be removed.

An interesting example is furnished by the equation

$$c = u(x) - xu'(x) + \frac{x^2}{2!} u''(x) - \frac{x^3}{3!} u'''(x) + \cdots,$$

where c is a constant, which has for its solution every analytic function satisfying the condition $u(0) = c$.

5. Solution by the "Reduction Principle." In general, however, it will be difficult to compute directly the zeros of the auxiliary function. If they are finite in number, they must necessarily correspond in number to the zeros of the generatrix function. In many cases where these roots are real, and even in some cases where they are complex, the so called "reduction principle" (principe de réduites) of the theory of linear equations in an infinite number of variables can be used to make this computation. A discussion of the theoretical basis of this principle is beyond the scope of this paper, but results derived by its use can always be tested in the original equation. For further information the reader is referred to F. Riesz: *Les Systèmes d'équations linéaires a une infinité d'inconnues*, Paris (1913), Chapter I.

This method consists in solving the system, composed of the homogeneous equation and its derived equations, for $\phi'(x)$ in terms of $\phi(x)$ by a succession of approximations, using first two equations in the unknowns $\phi'(x)$ and $\phi''(x)$, then three equations in the unknowns $\phi'(x)$, $\phi''(x)$, and $\phi'''(x)$, etc. If these approximations approach a limit as n is indefinitely increased, then the resulting equation of first order may be solved for ϕ.

Thus referring to system (4), in which $F(x)$ is to be replaced by zero, we have

for these successive approximations the equations:

$$x\phi_0' = -\frac{a_0}{\Delta_1}\phi_0, \qquad \phi_0 = cx^{-a_0/\Delta_1},$$

$$x\phi_1' = -\frac{a_0\delta_1}{\Delta_2}\phi_1, \qquad \phi_1 = cx^{-a_0\delta_1/\Delta_2},$$

. . . . . .

$$x\phi_n' = -\frac{a_0\delta_n}{\Delta_{n+1}}\phi_n, \qquad \phi_n = cx^{-a_0(\delta_n/\Delta_{n+1})},$$

where the δ_i are the leading principal minors of the negative co-factor of a_1 in the determinant of the constant coefficients of (4) and Δ_i has the same meaning as in section 2.

In case $\lim_{n=\infty} \frac{\delta_n}{\Delta_{n+1}}$ does not exist, a derived equation of second order can perhaps be found by removing both ϕ and ϕ' to the right-hand side of the equations in system (4) and solving for $\phi''(x)$. It is obvious that this reduction can not, in general, be carried on indefinitely since the order of an equation equivalent to system (4) can not exceed the number of zeros of the generatrix function.

As an example consider the homogeneous equation obtained by replacing the left-hand member of equation (11) by zero. By the reduction principle we obtain successively

$$\phi_0 = cx^{-2}, \qquad \phi_1 = cx^{1.6}, \qquad \phi_2 = cx^{2.23};$$

the last function is seen to be an approximation to the correct solution $\phi = x^{2.081}$.

In order to find the second solution we may reduce equation (11) to one of second order by solving system (4) for $\phi''(x)$. The first three approximations are easily found to be the following:

$$x^2\phi_0''(x) - \tfrac{2}{3}x\phi_0'(x) - \tfrac{4}{3}\phi_0 = 0,$$
$$x^2\phi_1''(x) - .323x\phi_1' - 1.883\phi_1 = 0,$$
$$x^2\phi_2''(x) - .112x\phi_2' - 2.312\phi_2 = 0.$$

The solutions of the last equation are found to be $\phi_2 = x^{2.175}$ and $\phi_2 = x^{-1.063}$, which are seen to approximate the correct solutions $\phi = x^{2.081}$ and $\phi = x^{-1.081}$.

THE GENERALIZED KRONECKER SYMBOL AND ITS APPLICATION TO THE THEORY OF DETERMINANTS.[1]

By F. D. MURNAGHAN, Johns Hopkins University.

1. Definition of the Generalized Kronecker Symbol. There is a symbol of frequent occurrence in algebra, and its applications to geometry, known as Kronecker's symbol. It is defined by the following properties. Let r and s be

[1] Read before the Md.-Va.-D. C. Section of the Association, May 10, 1924.

two labels to which we may assign independently any one of the n integral values $1, 2, 3 \cdots n$; then Kronecker's symbol[1] $\begin{bmatrix} r \\ s \end{bmatrix}$ has n^2 values as the n numerical values are assigned independently to the labels r and s. (The label r is here used as a superscript, and s as a subscript, but this not essential; it is, however, convenient in certain applications.) These n values are, by definition of the symbol, as follows:

$$\begin{bmatrix} r \\ s \end{bmatrix} = 0; \quad \text{if } r \neq s, \qquad \begin{bmatrix} r \\ s \end{bmatrix} = 1; \quad \text{if } r = s.$$

We shall now introduce a generalization of this symbol which we shall call the generalized Kronecker symbol and shall denote by $\begin{bmatrix} r_1 r_2 \cdots r_m \\ s_1 s_2 \cdots s_m \end{bmatrix}$; the integer m being any one of the n integers $1, 2, \cdots n$ and the labels $(r_1, r_2, \cdots r_m)$, $(s_1, s_2, \cdots s_m)$ being capable of taking, independently, any one of the n integral values. The symbol has then n^{2m} values and these are by definition as follows:

(*a*) $\begin{bmatrix} r_1 r_2 \cdots r_m \\ s_1 s_2 \cdots s_m \end{bmatrix} = 0$ if the set $(r_1, r_2, \cdots r_m)$ of m out of the n numbers $(1, 2, 3, \cdots n)$ is not the same as the set $(s_1, s_2, \cdots s_m)$ of m out of the same n numbers. Thus if $m = 2$, $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = 0$.

(*b*) $\begin{bmatrix} r_1 r_2 \cdots r_m \\ s_1 s_2 \cdots s_m \end{bmatrix}$ is alternating in both the superscripts and the subscripts. In other words a change in position of any two of the m superscripts (or subscripts) changes the sign but not the numerical value taken by the symbol. Thus $\begin{bmatrix} 1 & 2 \\ s_1 & s_2 \end{bmatrix} = -\begin{bmatrix} 2 & 1 \\ s_1 & s_2 \end{bmatrix}$ and $\begin{bmatrix} r_1 r_2 \\ 1 \quad 2 \end{bmatrix} = -\begin{bmatrix} r_1 r_2 \\ 2 \quad 1 \end{bmatrix}$. As an immediate consequence we have the result that $\begin{bmatrix} r_1 r_2 \cdots r_m \\ s_1 s_2 \cdots s_m \end{bmatrix} = 0$ if the numerical values taken by any two of the superscripts (or subscripts) are the same. *E.g.*, $\begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix} = 0$ and $\begin{bmatrix} r_1 r_2 \\ 1 \quad 1 \end{bmatrix} = 0$.

(*c*) When $(r_1, r_2, \cdots r_m)$ and $(s_1, s_2, \cdots s_m)$ are merely arrangements of the same set of m distinct numbers out of the set $(1, 2, \cdots n)$, the symbol $\begin{bmatrix} r_1 r_2 \cdots r_m \\ s_1 s_2 \cdots s_m \end{bmatrix}$ is to be assigned the value 1 if it requires an even number, and the value -1 if it requires an odd number, of inversions to get from the arrangement $(r_1, r_2, \cdots r_m)$ to the arrangement $(s_1, s_2, \cdots s_m)$. The three defining properties (*a*), (*b*), (*c*) of the symbol $\begin{bmatrix} r_1 r_2 \cdots r_m \\ s_1 s_2 \cdots s_m \end{bmatrix}$ may be resumed briefly as

[1] The usual notation is δ_s^r but the notation adopted here has, for the generalized symbol at any rate, distinct advantages from the typographical point of view.

follows: $\begin{bmatrix} r_1 r_2 \cdots r_m \\ s_1 s_2 \cdots s_m \end{bmatrix}$ is to be assigned the value zero unless $(r_1, r_2, \cdots r_m)$ and $(s_1, s_2, \cdots s_m)$ are arrangements of the same set of m distinct numbers out of the set $(1, 2, \cdots n)$. In this event it is to be assigned the value ± 1 according as these arrangements are of the same class or not.

2. Properties of the Generalized Kronecker Symbol. It will be convenient in stating these properties to adopt the convention that a *Greek* letter in an expression plays the rôle of a dummy or umbral symbol, *i.e.*, it is to be assigned in turn the n numerical values $(1, 2, \cdots n)$ and the n resulting expressions are to be added up. It will appear that, in the notation adopted, a Greek or summation letter will always appear twice in an expression, once as a subscript and once as a superscript.

It is an immediate consequence of the definition that

$$\begin{bmatrix} r_1 r_2 \cdots r_m \\ s_1 s_2 \cdots s_m \end{bmatrix} = \begin{bmatrix} s_1 s_2 \cdots s_m \\ r_1 r_2 \cdots r_m \end{bmatrix} \tag{2.1}$$

and also that

$$\begin{bmatrix} r_1 r_2 \cdots r_m \\ s_1 s_2 \cdots s_m \end{bmatrix} = \begin{bmatrix} r_1 r_2 \cdots r_m \\ \alpha_1 \alpha_2 \cdots \alpha_m \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ s_1 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ s_2 \end{bmatrix} \cdots \begin{bmatrix} \alpha_m \\ s_m \end{bmatrix}. \tag{2.2}$$

The expression on the right indicates, according to our convention, m distinct summations. But $\begin{bmatrix} \alpha_1 \\ s_1 \end{bmatrix} = 0$ unless the summation symbol α_1 is assigned the particular value s_1 so that the summation with respect to α_1 yields

$$\begin{bmatrix} r_1 r_2 \cdots r_m \\ s_1 \alpha_2 \cdots \alpha_m \end{bmatrix} \cdot \begin{bmatrix} \alpha_2 \\ s_2 \end{bmatrix} \cdots \begin{bmatrix} \alpha_m \\ s_m \end{bmatrix},$$

there being now $(m - 1)$ summations remaining. Continuing this argument we arrive at the result (2.2).

A similar argument gives the result

$$\begin{bmatrix} r_1 r_2 \cdots r_m \\ s_1 s_2 \cdots s_m \end{bmatrix} = \frac{1}{m!} \begin{bmatrix} r_1 r_2 \cdots r_m \\ \alpha_1 \alpha_2 \cdots \alpha_m \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \alpha_2 \cdots \alpha_m \\ s_1 s_2 \cdots s_m \end{bmatrix}. \tag{2.3}$$

For in the m summations on the right all the terms, for which $(\alpha_1, \alpha_2, \cdots \alpha_m)$ is not merely a rearrangement of *both* the groups of m distinct numbers $(r_1, r_2, \cdots r_m)$ and $(s_1, s_2, \cdots s_m)$, vanish. All the terms for which $(\alpha_1, \alpha_2, \cdots \alpha_m)$ is such an arrangement have the same value ± 1; for an inversion in any particular arrangement $(a_1, a_2, \cdots a_m)$ of the summation symbols $(\alpha_1, \alpha_2, \cdots \alpha_m)$ changes the sign of each of the factors $\begin{bmatrix} r_1 r_2 \cdots r_m \\ a_1 a_2 \cdots a_m \end{bmatrix}$ and $\begin{bmatrix} a_1 a_2 \cdots a_m \\ s_1 s_2 \cdots s_m \end{bmatrix}$.

Since there are precisely $m!$ such arrangements $(a_1, a_2, \cdots a_m)$ when the groups $(r_1, r_2, \cdots r_m)$ and $(s_1, s_2, \cdots s_m)$ of m distinct numbers are the same, the truth of (2.3) immediately follows.

We may show in a similar way that

$$\begin{bmatrix} r_1 r_2 \cdots r_n \\ s_1 s_2 \cdots s_n \end{bmatrix} = \frac{1}{m!}\frac{1}{(n-m)!}\begin{bmatrix} \alpha_1 \alpha_2 \cdots \alpha_m \\ s_1 s_2 \cdots s_m \end{bmatrix}\begin{bmatrix} \alpha_{m+1} \cdots \alpha_n \\ s_{m+1} \cdots s_n \end{bmatrix}\begin{bmatrix} r_1 r_2 \cdots r_n \\ \alpha_1 \alpha_2 \cdots \alpha_n \end{bmatrix} \tag{2.4}$$

or

$$\begin{bmatrix} r_1 r_2 \cdots r_n \\ s_1 s_2 \cdots s_n \end{bmatrix} = \frac{1}{m!}\frac{1}{(n-m)!}\begin{bmatrix} r_1 r_2 \cdots r_m \\ \alpha_1 \alpha_2 \cdots \alpha_m \end{bmatrix}\begin{bmatrix} r_{m+1} \cdots r_n \\ \alpha_{m+1} \cdots \alpha_n \end{bmatrix}\begin{bmatrix} \alpha_1 \alpha_2 \cdots \alpha_n \\ s_1 s_2 \cdots s_n \end{bmatrix}. \tag{2.4^{bis}}$$

First of all it is evident that in the summations on the right-hand side of (2.4) $(\alpha_1, \alpha_2, \cdots \alpha_m)$, $(\alpha_{m+1}, \cdots \alpha_n)$ must be arrangements of $(s_1, s_2, \cdots s_m)$ and $(s_{m+1}, \cdots s_n)$ respectively and $(\alpha_1, \alpha_2, \cdots \alpha_n)$ must be an arrangement of $(1, 2, \cdots n)$ for the corresponding term to have a value different from zero. Hence $(s_1, s_2, \cdots s_n)$ must be an arrangement of the n numbers $(1, 2, \cdots n)$ for the expression on the right of (2.4) to have a non-zero value. An inversion in any arrangement $(a_1, a_2, \cdots a_m)$ of $(\alpha_1, \alpha_2, \cdots \alpha_m)$ changes the sign of both the factors $\begin{bmatrix} a_1 a_2 \cdots a_m \\ s_1 s_2 \cdots s_m \end{bmatrix}$ and $\begin{bmatrix} r_1 r_2 \cdots r_n \\ a_1 a_2 \cdots a_n \end{bmatrix}$ of the corresponding term in the summation and so the value of the term itself is unaltered. This explains the presence of the numerical factor $1/m!\,(n-m)!$.

3. The Definition of a Determinant. Let us consider the n quantities a_s^r where r and s run independently over the n numbers $(1, 2, \cdots n)$. We shall define the determinant of these n quantities as

$$D = \begin{bmatrix} 1 & 2 & \cdots & n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \tag{3.1}$$

and this determinant will be said to be of the nth order. It is immediately apparent that if $(r_1, r_2, \cdots r_n)$ is any arrangement of the n numbers $(1, 2, \cdots n)$, we may write D in the equivalent form

$$D = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} a_{r_1}^{\alpha_1} a_{r_2}^{\alpha_2} \cdots a_{r_n}^{\alpha_n}. \tag{3.2}$$

For an inversion in the arrangement $(r_1, r_2, \cdots r_n)$ carries with it an inversion in the arrangement $(\alpha_1, \alpha_2, \cdots \alpha_n)$. We may, for example, in the case $n = 3$, write D or $\begin{bmatrix} 1 & 2 & 3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3}$ in the equivalent form $\begin{bmatrix} 2 & 1 & 3 \\ \alpha_2 & \alpha_1 & \alpha_3 \end{bmatrix} a_2^{\alpha_2} a_1^{\alpha_1} a_3^{\alpha_3}$ and this, by a mere interchange of the summation symbols α_1 and α_2, is $\begin{bmatrix} 2 & 1 & 3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} a_2^{\alpha_1} a_1^{\alpha_2} a_3^{\alpha_3}$. We derive from (3.2) upon giving to the n labels $(r_1, r_2, \cdots r_n)$, each independently, the n values $(1, 2, \cdots n)$ and then summing the

resulting expressions the result

$$D = \frac{1}{n!}\begin{bmatrix} \rho_1 & \rho_2 & \cdots & \rho_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} a_{\rho_1}^{\alpha_1} a_{\rho_2}^{\alpha_2} \cdots a_{\rho_n}^{\alpha_n}. \tag{3.3}$$

This result is important as it shows that the superscripts and subscripts of the n^2 elements a_s^r of the determinant play exactly the same rôles. In fact the determinant D might just as well have been defined by the equation

$$D' = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 1 & 2 & \cdots & n \end{bmatrix} a_{\alpha_1}^1 a_{\alpha_2}^2 \cdots a_{\alpha_n}^n \tag{3.1^{bis}}$$

as by the equation (3.1). For we derive as before

$$D' = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ r_1 & r_2 & \cdots & r_n \end{bmatrix} a_{\alpha_1}^{r_1} a_{\alpha_2}^{r_2} \cdots a_{\alpha_n}^{r_n} \tag{3.2^{bis}}$$

$$= \frac{1}{n!}\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \rho_1 & \rho_2 & \cdots & \rho_n \end{bmatrix} a_{\alpha_1}^{\rho_1} a_{\alpha_2}^{\rho_2} \cdots a_{\alpha_n}^{\rho_n}. \tag{3.3^{bis}}$$

Whence $D' = D$, *i.e.*, the value of the determinant of n^2 ordered elements a_s^r is unaltered by the change of order corresponding to the interchange of subscript and superscript. Arranging the n^2 elements into n rows of n elements each in the usual way, this is the result that an interchange of rows and columns does not affect the value of a determinant.

It is somewhat more symmetrical to define the quantity $D^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$ by the equation

$$D^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} a_{s_1}^{\alpha_1} a_{s_2}^{\alpha_2} \cdots a_{s_n}^{\alpha_n}. \tag{3.4}$$

Then $D^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$ is alternating in both the superscripts $(r_1, r_2, \cdots r_n)$ and the subscripts $(s_1, s_2, \cdots s_n)$. It has, accordingly, the value zero unless $(r_1, r_2, \cdots r_n)$ and $(s_1, s_2, \cdots s_n)$ are merely arrangements of the n distinct numbers $(1, 2, \cdots n)$. In this event $D^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \pm D$ according as these arrangements are of the same class or not. Just as before we have a second expression for $D^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n}$, namely

$$D^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ s_1 & s_2 & \cdots & s_n \end{bmatrix} a_{\alpha_1}^{r_1} a_{\alpha_2}^{r_2} \cdots a_{\alpha_n}^{r_n}. \tag{3.4^{bis}}$$

It follows from (3.4) and (2.2) that the generalized Kronecker symbol $\begin{bmatrix} r_1 & r_2 & \cdots & r_m \\ s_1 & s_2 & \cdots & s_m \end{bmatrix}$ is a determinant of order m of which the elements are values of the ordinary Kronecker symbol $\begin{bmatrix} r \\ s \end{bmatrix}$. Thus

$$\begin{bmatrix} r_1 & r_2 \\ s_1 & s_2 \end{bmatrix} = \begin{vmatrix} \begin{bmatrix} r_1 \\ s_1 \end{bmatrix} & \begin{bmatrix} r_1 \\ s_2 \end{bmatrix} \\ \begin{bmatrix} r_2 \\ s_1 \end{bmatrix} & \begin{bmatrix} r_2 \\ s_2 \end{bmatrix} \end{vmatrix}, \qquad \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{bmatrix} = \begin{vmatrix} \begin{bmatrix} r_1 \\ s_1 \end{bmatrix} & \begin{bmatrix} r_1 \\ s_2 \end{bmatrix} & \begin{bmatrix} r_1 \\ s_3 \end{bmatrix} \\ \begin{bmatrix} r_2 \\ s_1 \end{bmatrix} & \begin{bmatrix} r_2 \\ s_2 \end{bmatrix} & \begin{bmatrix} r_2 \\ s_3 \end{bmatrix} \\ \begin{bmatrix} r_3 \\ s_1 \end{bmatrix} & \begin{bmatrix} r_3 \\ s_2 \end{bmatrix} & \begin{bmatrix} r_3 \\ s_3 \end{bmatrix} \end{vmatrix}$$

and so on.

4. Laplace's Expansion of a Determinant. If we use the result (2.4) in (3.4), we obtain

$$\begin{aligned} D^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} &= \frac{1}{m!(n-m)!} \begin{bmatrix} \beta_1 \beta_2 \cdots \beta_m \\ \alpha_1 \alpha_2 \cdots \alpha_m \end{bmatrix} \begin{bmatrix} \beta_{m+1} \cdots \beta_n \\ \alpha_{m+1} \cdots \alpha_n \end{bmatrix} \begin{bmatrix} r_1 \cdots r_n \\ \beta_1 \cdots \beta_n \end{bmatrix} a^{\alpha_1}_{s_1} a^{\alpha_2}_{s_2} \cdots a^{\alpha_n}_{s_n} \\ &= \frac{1}{m!(n-m)!} \begin{bmatrix} r_1 r_2 \cdots r_n \\ \beta_1 \beta_2 \cdots \beta_n \end{bmatrix} \left\{ \begin{bmatrix} \beta_1 \cdots \beta_m \\ \alpha_1 \cdots \alpha_m \end{bmatrix} a^{\alpha_1}_{s_1} \cdots a^{\alpha_m}_{s_m} \right\} \\ &\qquad \times \left\{ \begin{bmatrix} \beta_{m+1} \cdots \beta_n \\ \alpha_{m+1} \cdots \alpha_n \end{bmatrix} a^{\alpha_{m+1}}_{s_{m+1}} \cdots a^{\alpha_n}_{s_n} \right\}, \end{aligned} \tag{4.1}$$

where the face brackets indicate that the summations are performed first with respect to $(\alpha_{m+1}, \cdots \alpha_n)$, then with respect to $(\alpha_1, \cdots \alpha_m)$, and finally with respect to $(\beta_1, \cdots \beta_n)$. Although $(\alpha_{m+1}, \cdots \alpha_n)$ are to run over the n values $(1, 2, \cdots n)$, they may be regarded as running merely over the $(n - m)$ distinct values $(b_{m+1}, \cdots b_n)$ in the summation

$$\begin{bmatrix} b_{m+1} \cdots b_n \\ \alpha_{m+1} \cdots \alpha_n \end{bmatrix} a^{\alpha_{m+1}}_{s_{m+1}} \cdots a^{\alpha_n}_{s_n} \tag{4.2}$$

as the terms of this summation in which any one of the umbral symbols is assigned a value not in the set $(b_{m+1}, \cdots b_n)$ will all be zero from the definition of the symbol $\begin{bmatrix} b_{m+1}, \cdots b_n \\ \alpha_{m+1}, \cdots \alpha_n \end{bmatrix}$. The summation (4.2) is, therefore, a determinant of order $(n - m)$ which may conveniently be denoted by $D^{b_{m+1} \cdots b_n}_{s_{m+1} \cdots s_n}$. Dealing similarly with the second set of summations in (4.1), we obtain

$$D^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \frac{1}{m!(n-m)!} \begin{bmatrix} r_1 r_2 \cdots r_n \\ \beta_1 \beta_2 \cdots \beta_n \end{bmatrix} D^{\beta_1 \cdots \beta_m}_{s_1 \cdots s_m} D^{\beta_{m+1} \cdots \beta_n}_{s_{m+1} \cdots s_n} \tag{4.3}$$

which is Laplace's expansion. We may avoid the numerical factor $1/m!\,(n - m)!$ by agreeing that instead of assigning to the summation symbols β the values 1 to n independently we understand that $(\beta_1, \cdots \beta_m)$ is a group of m out of the n symbols $(r_1, r_2, \cdots r_n)$, $(\beta_{m+1}, \cdots \beta_n)$ being the group of $(n - m)$ that is left, and that the same group is not repeated. We have then, for example,

$$D = D^{r_1 r_2 \cdots r_n}_{r_1 r_2 \cdots r_n} = \sum \begin{bmatrix} r_1 \cdots r_n \\ \beta_1 \cdots \beta_n \end{bmatrix} D^{\beta_1 \cdots \beta_m}_{r_1 \cdots r_m} D^{\beta_{m+1} \cdots \beta_n}_{r_{m+1} \cdots r_n} \tag{4.4}$$

there being $\binom{n}{m}$ terms in the summation on the right. In the special case when $m = 1$ we obtain the usual expansion in terms of any one row. It may be remarked that, owing to the symmetrical way in which the subscripts and superscripts enter the definition of a determinant, we have the alternative expression for $D^{r_1 \cdots r_n}_{s_1 \cdots s_n}$

$$D^{r_1 r_2 \cdots r_n}_{s_1 s_2 \cdots s_n} = \frac{1}{m!\,(n-m)!} \begin{bmatrix} \beta_1 \cdots \beta_n \\ s_1 \cdots s_n \end{bmatrix} D^{r_1 \cdots r_m}_{\beta_1 \cdots \beta_m} D^{r_{m+1} \cdots r_n}_{\beta_{m+1} \cdots \beta_n}. \qquad (4.3^{bis})$$

5. The Multiplication of Determinants. Let us consider two determinants D and $\overline{D}$ of order n whose elements may be denoted by a_s^r and b_s^r respectively. We have on multiplication of the two expressions

$$D^{r_1 \cdots r_n}_{s_1 \cdots s_n} = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} a^{\alpha_1}_{s_1} \cdots a^{\alpha_n}_{s_n} \qquad (Cf.\ 3.4)$$

$$\overline{D}^{s_1 \cdots s_n}_{r_1 \cdots r_n} = \begin{bmatrix} \beta_1 \cdots \beta_n \\ r_1 \cdots r_n \end{bmatrix} b^{s_1}_{\beta_1} \cdots b^{s_n}_{\beta_n} \qquad (Cf.\ 3.4^{bis})$$

by one another the result

$$D^{r_1 \cdots r_n}_{s_1 \cdots s_n} \cdot \overline{D}^{s_1 \cdots s_n}_{r_1 \cdots r_n} = \begin{bmatrix} \beta_1 \cdots \beta_n \\ r_1 \cdots r_n \end{bmatrix} \begin{bmatrix} r_1 \cdots r_n \\ \alpha_1 \cdots \alpha_n \end{bmatrix} a^{\alpha_1}_{s_1} b^{s_1}_{\beta_1} \cdots a^{\alpha_n}_{s_n} b^{s_n}_{\beta_n}. \qquad (5.1)$$

This product is zero unless $(s_1, s_2, \cdots s_n)$ is an arrangement of the n numbers $(1, 2, \cdots n)$ and it is unaltered by any change in this arrangement for an inversion of any two of the labels $(s_1, s_2, \cdots s_n)$ merely changes the sign of both factors $D^{r_1 \cdots r_n}_{s_1 \cdots s_n}$ and $\overline{D}^{s_1 \cdots s_n}_{r_1 \cdots r_n}$. We may therefore give $(s_1, s_2, \cdots s_n)$, independently, the n values $(1, 2, \cdots n)$ and sum up the resulting expressions when we find

$$D^{r_1 \cdots r_n}_{\sigma_1 \cdots \sigma_n} \cdot \overline{D}^{\sigma_1 \cdots \sigma_n}_{r_1 \cdots r_n} = \begin{bmatrix} r_1 \cdots r_n \\ \alpha_1 \cdots \alpha_n \end{bmatrix} \begin{bmatrix} \beta_1 \cdots \beta_n \\ r_1 \cdots r_n \end{bmatrix} c^{\alpha_1}_{\beta_1} \cdots c^{\alpha_n}_{\beta_n}, \qquad (5.2)$$

where

$$c_s^r = a_\sigma^r b_s^\sigma. \qquad (5.3)$$

Just as the expression (5.1) was unaltered by a rearrangement of the labels $(s_1, s_2, \cdots s_n)$, it is unaltered by a rearrangement of the labels $(r_1, r_2, \cdots r_n)$. We have, then, on denoting $D^{1\,2 \cdots n}_{1\,2 \cdots n}$ and $\overline{D}^{1\,2 \cdots n}_{1\,2 \cdots n}$ simply by D and $\overline{D}$, respectively, as before

$$D\,\overline{D} = D^{r_1 \cdots r_n}_{s_1 \cdots s_n}\ \overline{D}^{s_1 \cdots s_n}_{r_1 \cdots r_n} \qquad \{(r_1, r_2, \cdots r_n) \text{ and } (s_1, s_2, \cdots s_n) \text{ being arrangements of the } n \text{ numbers } (1, 2, \cdots n)\}$$

$$= \frac{1}{n!} D^{r_1 r_2 \cdots r_n}_{\sigma_1 \sigma_2 \cdots \sigma_n}\ \overline{D}^{\sigma_1 \cdots \sigma_n}_{r_1 \cdots r_n}$$

$$= \left(\frac{1}{n!}\right)^2 \cdot D^{\rho_1 \cdots \rho_n}_{\sigma_1 \cdots \sigma_n}\ \overline{D}^{\sigma_1 \cdots \sigma_n}_{\rho_1 \cdots \rho_n},$$

where the Greek letters are as always summation symbols. Hence the product $D\overline{D}$ of the two determinants of order n is given by

$$D\overline{D} = \left(\frac{1}{n!}\right)^2 \left\{ \begin{bmatrix} \rho_1 & \cdots & \rho_n \\ \alpha_1 & \cdots & \alpha_n \end{bmatrix} \begin{bmatrix} \beta_1 & \cdots & \beta_n \\ \rho_1 & \cdots & \rho_n \end{bmatrix} \right\} c_{\beta_1}^{\alpha_1} \cdots c_{\beta_n}^{\alpha_n}$$

$$= \frac{1}{n!} \begin{bmatrix} \beta_1 & \cdots & \beta_n \\ \alpha_1 & \cdots & \alpha_n \end{bmatrix} c_{\beta_1}^{\alpha_1} \cdots c_{\beta_n}^{\alpha_n}, \qquad \text{by (2.3).}$$

It is accordingly, by (3.3), the determinant of the elements c_s^r defined by (5.3). This is the usual law of the multiplication of determinants where we multiply "rows into columns."

If we glance again at the equation (5.1), we may derive at once a theorem of frequent application in algebra. Thus we have

$$D_{s_1 s_2 \cdots s_n}^{1\,2 \cdots n} \cdot \overline{D}_{1\,2 \cdots n}^{s_1 s_2 \cdots s_n} = \frac{1}{n!} D_{s_1 \cdots s_n}^{\rho_1 \cdots \rho_n} \overline{D}_{\rho_1 \cdots \rho_n}^{s_1 \cdots s_n}$$

$$= \begin{bmatrix} \beta_1 & \cdots & \beta_n \\ \alpha_1 & \cdots & \alpha_n \end{bmatrix} (a_{s_1}^{\alpha_1} b_{\beta_1}^{s_1}) \cdots (a_{s_n}^{\alpha_n} b_{\beta_n}^{s_n})$$

as before. We may now let the labels $(s_1, s_2, \cdots s_n)$ run over a set of integers $(1, 2, \cdots p)$, where $p > n$, and we obtain on the left a summation of products of determinants of the nth order and on the right a determinant of the nth order whose elements are $c_s^r = a_\sigma^r b_s^\sigma$ (the summation symbol σ running over the numbers $1, 2, \cdots p$). Thus

$$D_{\sigma_1 \sigma_2 \cdots \sigma_n}^{1\,2 \cdots n} \cdot \overline{D}_{1 \cdots n}^{\sigma_1 \cdots \sigma_n} = \begin{bmatrix} \beta_1 & \cdots & \beta_n \\ \alpha_1 & \cdots & \alpha_n \end{bmatrix} c_{\beta_1}^{\alpha_1} \cdots c_{\beta_n}^{\alpha_n}.$$

In the summation on the left a term $D_{s_1 s_2 \cdots s_n}^{1\,2 \cdots n} \overline{D}_{1\,2 \cdots n}^{s_1 s_2 \cdots s_n}$ is repeated $n!$ times and we may state our result as follows:

Given two sets of np ordered elements each a_i^r, and b_s^t, respectively $(r, s = 1, 2, \cdots n;\ t = 1, 2, \cdots p)$, $p > n$, the determinant of order n whose elements c_s^r are given by $c_s^r = a_\tau^r b_s^\tau$ is equal to $\sum D_{\tau_1 \tau_2 \cdots \tau_n}^{1\,2 \cdots n} \overline{D}_{1\,2 \cdots n}^{\tau_1 \cdots \tau_n}$ where $(\tau_1, \cdots \tau_n)$ is any set of n distinct numbers out of the p numbers $(1, 2, \cdots p)$ and no set is repeated in the summation.

6. Conclusion. In the foregoing paragraphs we have endeavored to give a presentation of determinants in which the fundamental theorems would follow naturally from the definition. In the usual presentation the emphasis is laid, not on the explicit expression of a determinant as a function of its elements, but on the expansion theorem. This has obvious advantages in practical work with determinants but it seems to lack advantages for the theoretical discussion. Whilst the proof of such theorems as Laplace's expansion and the product theorem is not difficult, it has more the appearance of a verification of a known result than of the derivation of a new result. We hope to have shown above

that the consistent use of the explicit expression of a determinant as a function of its elements, given in (3.1), leads naturally to the theorems mentioned.

It may be pointed out that Kronecker [1] defined a determinant by means of the following characteristic properties:

(*a*) It is a function of the n^2 elements a_s^r (which we may suppose arranged in n rows of n elements each), which is linear and homogeneous in the elements of each row.

(*b*) It changes its sign when any two rows are interchanged.

(*c*) It has the value unity when the elements a_s^r take the values of Kronecker's symbol $\left[\begin{matrix} r \\ s \end{matrix}\right]$.

This postulational definition leads to the explicit expression (3.1). Again, however, the proof of the law of multiplication by means of these postulates has more the aspect of a verification than a derivation. The novelty in the present presentation lies essentially in the symmetrical use of subscripts and superscripts. Usually when an explicit formula for a determinant has been given in the past, a symbol $\epsilon_{r_1 \cdots r_n}$, which is essentially equivalent to $\left[\begin{matrix} 1 & 2 & \cdots & n \\ r_1 & r_2 & \cdots & r_n \end{matrix}\right]$, has been used. As we hope to show elsewhere, the symbol $\left[\begin{matrix} r_1 & r_2 & \cdots & r_n \\ s_1 & s_2 & \cdots & s_n \end{matrix}\right]$ seems to be really more fundamental than the symbol $\epsilon_{r_1 \cdots r_n}$.[2]

THE CIRCULAR CUBIC ON TWENTY-ONE POINTS OF A TRIANGLE.

By T. W. MOORE and J. H. NEELLEY, Graduate Students, Yale University.

1. Introduction. This article is concerned with the circular cubic on twenty-one notable points of a general triangle, first mentioned by Neuberg and referred to by Professor B. H. Brown in his article entitled "A Theorem on Isogonal Tetrahedra." [3]

While the theory of the circular cubic is well known and much has been written on it,[4] the equation of this particular curve as derived is of considerable interest and the results appear in particularly neat form. To serve our purpose, we shall take the given triangle as a triangle of reference and use a system of notation practically the same as that used by Clebsch in his article on the geometry of the triangle.[5]

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[1] Leopold Kronecker, *Vorlesungen über aie Theorie der Determinanten*; bearbeitet von K. Hensel; Erster Band, p. 291 et seq.

[2] Since writing the above paper we have read a paper by E. H. Moore, "A fundamental remark concerning Determinantal Notations," *Annals of Mathematics*, ser. 2, vol. 1, p. 177, which should be consulted by those interested in the subject of this paper.

[3] This MONTHLY, *1924*, 371–375. See also the note by Professor Brown on page 247 of this issue. EDITOR.

[4] Casey, *Transactions of the Royal Irish Academy*, volume 24, 1871. Loria, *Spezielle Algebraische und Transcendente Ebene Kurven*, pp. 31–35. Basset, *Elementary Treatise on Cubic and Quartic Curves*, chapters VI and IX. Hilton, *Plane Algebraic Curves*, chapter XIV. *Encyklopädie der Mathematischen Wissenschaften*, Band 3, Teil 2, 1ste Hälfte, p. 510.

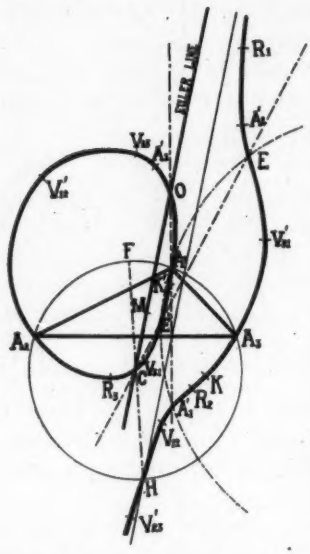
[5] Clebsch-Lindemann, *Vorlesungen über Geometrie*, pp. 312 ff.

In this notation the coördinates of points and lines and the coefficients in the equations involved are expressed as trigonometric functions of three angles ϕ_i, $i = 1, 2, 3$. Two of these angles are interior angles of the triangle and the remaining one is the opposite exterior angle given a negative sign, so that $\phi_1 + \phi_2 + \phi_3 = 0$. With Clebsch we shall place $s_i = \sin \phi_i$, $c_i = \cos \phi_i$ and use the symbols $\sigma_1, \sigma_2, \sigma_3$ for the sines of $\phi_2 - \phi_3$, $\phi_3 - \phi_1$, and $\phi_1 - \phi_2$ respectively.

2. The Equation of the Neuberg Cubic. In order to get the equation of the Neuberg cubic, we recall that the equation of any circular cubic may be written in the form

$$U \equiv QL_\infty + \lambda C_1 L_1 = 0, \tag{1}$$

where Q is a conic, L_∞ the line at infinity, C_1 a circle, and L_1 a line. Of the twenty-one points mentioned by Neuberg let us make use of the group given below:



(a) the vertices of the reference triangle, $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, $A_3(0, 0, 1)$,

(b) the reflections of the vertices in the opposite sides, A_1', A_2', A_3' respectively,

(c) the circumcenter C of the base triangle,

(d) the orthocenter O of the base triangle,

(e) the isodynamic centers, or equilateral poles, E and E', the two points of intersection of the three circles of Apollonius.

Together with these points which can be readily constructed, we shall use the circular points I and J.

All these points are mentioned by Clebsch except A_1', A_2', A_3'. The coordinates of C and O are (c_1, c_2, c_3) and $(1/c_1, 1/c_2, 1/c_3)$ respectively. By use of harmonic properties we obtain the coördinates of the points A_1', A_2', A_3' as $A_1'(\frac{1}{2}, c_3, c_2)$, $A_2'(c_3, \frac{1}{2}, c_1)$, $A_3'(c_2, c_1, \frac{1}{2})$. The circular points I and J are found by solving the equations of the circumcircle and the ideal line and are $(c_2 \pm is_2, c_1 \mp is_1, 1)$.

The equations of the three circles of Apollonius [1] may be written in the form

$$\begin{aligned} x_2^2 - x_3^2 + 2c_2x_3x_1 - 2c_3x_1x_2 &= 0, \\ x_3^2 - x_1^2 + 2c_3x_1x_2 - 2c_1x_2x_3 &= 0, \\ x_1^2 - x_2^2 + 2c_1x_2x_3 - 2c_2x_3x_1 &= 0, \end{aligned}$$

and, from inspection, the equation of the common chord of these three circles is

$$(\sigma x) \equiv \sigma_1 x_1 + \sigma_2 x_2 + \sigma_3 x_3 = 0.$$

[1] The equations given in Clebsch-Lindemann, p. 319, are readily reduced to the forms given.

Now, of the eight points A_1, A_1', E, E', I, J, C, O, we see, either by definition or inspection, that the first six are on the Apollonius circle through the vertex A_1, and the remaining two are on the Euler line of the triangle, the line [1]

$$(sc\sigma x) \equiv s_1c_1\sigma_1x_1 + s_2c_2\sigma_2x_2 + s_3c_3\sigma_3x_3 = 0.$$

Again, three of these points, A_1, A_1' and O, are on the altitude through the vertex A_1, while the points E, E' and C are on the common chord of the three circles of Apollonius.

Hence the pencil of circular cubics on these eight points may be written in the form

$$L_2L_3L_\infty + \lambda C_1L_1 = 0, \tag{2}$$

where C_1 is the Apollonius circle through A_1, L_1 the Euler line, L_2 the altitude of the triangle on the vertex A_1, and L_3 the common chord of the circles of Apollonius. From the form of equation (2) it is obvious that the remaining point common to every member of the pencil is the intersection of the Euler line with the line at infinity. Therefore, we have the following

THEOREM: *Every member of the pencil of circular cubics on the eight points given above has its real asymptote parallel to the Euler line of the base triangle.*

To determine the Neuberg cubic itself from this pencil (2), the parameter λ must be such that the curve will pass through another vertex, say A_2. This means that the coefficient of x_2^3 in the expanded form of (2) must vanish. The equations of the various parts of (2) are:

$$\begin{aligned}
C_1 &\equiv x_2^2 - x_3^2 + 2c_2x_3x_1 - 2c_3x_1x_2 = 0,\\
L_1 &\equiv (sc\sigma x) \equiv s_1c_1\sigma_1x_1 + s_2c_2\sigma_2x_2 + s_3c_3\sigma_3x_3 = 0,\\
L_2 &\equiv c_2x_2 - c_3x_3 = 0,\\
L_3 &\equiv (\sigma x) \equiv \sigma_1x_1 + \sigma_2x_2 + \sigma_3x_3 = 0,\\
L_\infty &\equiv (sx) \equiv s_1x_1 + s_2x_2 + s_3x_3 = 0.
\end{aligned}$$

Now if the curve

$$(c_2x_2 - c_3x_3)(\sigma x)(sx) + \lambda(x_2^2 - x_3^2 + 2c_2x_3x_1 - 2c_3x_1x_2)(sc\sigma x) = 0 \tag{3}$$

is required to pass through $A_2(0, 1, 0)$, the result is $\lambda = -1$.

If equation (3) is expanded for this value of λ and simplified by trigonometric reductions, we find that the coefficient of the product term $x_1x_2x_3$ vanishes identically and that $s_1\sigma_1$ is a factor of the remaining terms. So the equation of the circular cubic of Neuberg reduces to the symmetrical form

$$(c_1+2c_2c_3)(x_2^2-x_3^2)x_1+(c_2+2c_3c_1)(x_3^2-x_1^2)x_2+(c_3+2c_1c_2)(x_1^2-x_2^2)x_3=0. \tag{4}$$

3. Points Associated with the Curve. Of the twenty-one points on the curve mentioned by Neuberg, or rather twenty-three, if we include the circular points,

[1] Clebsch-Lindemann, p. 334.

we have already the coördinates of ten. The coördinates of the isodynamic points E and E' are $(c_1\sqrt{3} \pm s_1,\ c_2\sqrt{3} \pm s_2,\ c_3\sqrt{3} \pm s_3)$. The other points mentioned by Neuberg are:

(a) the isogonal points, K and K', $[1/(c_1\sqrt{3}\pm s_1),\ 1/(c_2\sqrt{3}\pm s_2),\ 1/(c_3\sqrt{3}\pm s_3)]$, whose coördinates are the reciprocals of those of the isodynamic points,

(b) the three points R_i, symmetric to the vertices A_i in the lines $q_{jk}q_{kj}$, where q_{jk} is the point in which the side a_j of the reference triangle meets the perpendicular bisector of the side a_k. Their coördinates are $(2,\ 1/c_3,\ 1/c_2)$, $(1/c_3,\ 2,\ 1/c_1)$, $(1/c_2,\ 1/c_1,\ 2)$, or the reciprocal points to the reflections of the vertices in the opposite sides,

(c) the six vertices V_{ij} and V_{ij}' of the equilateral triangles formed on the sides of the base triangle are $(\sqrt{3},\ c_3\sqrt{3} \pm s_3,\ c_2\sqrt{3} \pm s_2)$, $(c_3\sqrt{3} \pm s_3,\ \sqrt{3},\ c_1\sqrt{3} \pm s_1)$, and $(c_2\sqrt{3} \pm s_2,\ c_1\sqrt{3} \pm s_1,\ \sqrt{3})$.

In addition to these, we have found a number of other points on the curve, which can be easily constructed, among which may be mentioned the remaining intersection of each side of the base triangle with the curve. These points are $(0,\ c_2 + 2c_3c_1,\ c_3 + 2c_1c_2)$, etc., and are the points where each side of the triangle meets the line through the opposite vertex parallel to the Euler line. Also the points $(1,\ \pm 1,\ \pm 1)$ are of particular interest as they are the centers of the inscribed and three escribed circles of the base triangle. Furthermore, the four points $(1,\ \pm 1,\ \pm 1)$ are the vertices of a complete quadrangle inscribed in the curve of which the vertices of the reference triangle are the diagonal points. (*Cf.* Clebsch-Lindemann, p. 327.)

If we write the equations of the tangents to the cubic at the points $(1,\ \pm 1,\ \pm 1)$, we find that they all meet in the point $(c_1 + 2c_2c_3,\ c_2 + 2c_3c_1,\ c_3 + 2c_1c_2)$, which is the point at infinity on the Euler line. Furthermore, the tangents to the cubic at the vertices of the base triangle, or the diagonal points of the quadrangle, meet on the cubic at the point $[1/(c_1 + 2c_2c_3),\ 1/(c_2 + 2c_3c_1),\ 1/(c_3 + 2c_1c_2)]$, the remaining point of intersection H of the cubic and the circumcircle.[1] Therefore the real asymptote of the curve is the line through this last point parallel to the Euler line. In addition the four points $(1,\ \pm 1,\ \pm 1)$ appear as the centers of inversion of this circular cubic.[2]

The double, or singular focus F of the curve is the opposite extremity of the diameter of the circumcircle $(s/x) = 0$ on the point where the asymptote cuts the circle and cubic,[3] and has coördinates $(1/\sigma_1,\ 1/\sigma_2,\ 1/\sigma_3)$. The circumcircle of the base triangle is the common Feuerbach circle of the four triangles formed by omitting in turn one of the points $(1,\ \pm 1,\ \pm 1)$.[4]

4. Special Cases of the Neuberg Cubic. If the base triangle becomes isosceles, with the vertex opposite the odd side at $(0,\ 0,\ 1)$, then $s_2 = s_1,\ s_3 = -2s_1c_1$, $c_2 = c_1$, and $c_3 = 2c_1^2 - 1$. The Apollonius circle through this vertex $A_3(0,\ 0,\ 1)$

[1] Salmon-Fiedler, *Höhere Ebene Kurven*, p. 168.
[2] Basset, *l.c.*, p. 157.
[3] Casey, *Transactions of the Royal Irish Academy*, vol. 24, p. 496.
[4] Basset, *Ibid.*, p. 158.

breaks up into $(x_1 - x_2)(x_1 + x_2 - 2c_1x_3) = 0$, representing the Euler line and the line at infinity. The cubic (4) reduces to the form

$$(x_1 - x_2)[c_1(x_3^2 + x_1x_2) - (x_1 + x_2)x_3] = 0,$$

which is the Euler line and a circle having its center at A_3, and passing through the remaining vertices of the reference triangle.

If the base triangle becomes equilateral, each coefficient of equation (4) vanishes identically, and as Professor Brown has pointed out, the cubic is not unique. Of the twenty-one points given by Neuberg, those which have determinate positions are located at one of the following seven points $(1, \pm 1, \pm 1)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and the others have arbitrary positions. These seven points, together with the two circular points, are the base points of a pencil of cubics which we shall consider.

5. Net of Cubics on Seven Points. While any four points in the plane, no three of which are collinear, may be given the projective coördinates $(1, \pm 1, \pm 1)$, it must be remembered that we are preserving metric properties and that the four centers of inversion of a circular cubic form an orthic set, that is, each point is the orthocenter of the triangle formed by the other three. In the notation used these four centers of inversion have coördinates $(1, \pm 1, \pm 1)$, the diagonal points being the reference points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, so any cubic on the seven points may be written in the form [1]

$$k_1(x_2^2 - x_3^2)x_1 + k_2(x_3^2 - x_1^2)x_2 + k_3(x_1^2 - x_2^2)x_3 = 0, \tag{6}$$

or

$$\begin{vmatrix} k_1 & k_2 & k_3 \\ x_1 & x_2 & x_3 \\ \frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_3} \end{vmatrix} = 0. \tag{7}$$

Obviously, if the cubic (7) passes through any point (m_1, m_2, m_3), it will also pass through the point $(1/m_1, 1/m_2, 1/m_3)$. Hence the pencil of cubics on the seven given points and any eighth point (m) will have the remaining common intersection at $(1/m)$, and the coefficients of any member of this pencil, in terms of k_i, will be the coördinates of a point on the line joining (m) and $(1/m)$. Therefore the Neuberg cubic may be considered as the circular cubic of that pencil of cubics in which two members have k_i equal to c_i and $1/c_i$ respectively. Since $(1/s)$, the point of intersection M of the medians, is also on the Euler line, this same pencil of cubics contains the seventeen-point cubic

$$\frac{1}{s_1}(x_2^2 - x_3^2)x_1 + \frac{1}{s_2}(x_3^2 - x_1^2)x_2 + \frac{1}{s_3}(x_1^2 - x_2^2)x_3 = 0 \tag{8}$$

mentioned by Casey.[2]

[1] *Cf.* Salmon-Fiedler, *Höhere Ebene Kurven*, p. 258.
[2] Casey, *Analytic Geometry*, p. 460.

The net of cubics (6) contains a pencil of circular cubics, the necessary and sufficient condition being $k_1s_1 + k_2s_2 + k_3s_3 = 0$. The cubics of this pencil are in an involution of pairs such that for each pair the real asymptotes are perpendicular to one another, and the double focus of each is the point where the other curve meets its asymptote. Thus the cubic (4) and the cubic

$$\sigma_1(x_2^2 - x_3^2)x_1 + \sigma_2(x_3^2 - x_1^2)x_2 + \sigma_3(x_1^2 - x_2^2)x_3 = 0 \qquad (9)$$

are a pair in this involution.

6. Self-Inversion of the Net of Cubics. From the equation (7) we see that each cubic of the net is inverted into itself by the quadratic transformation $y_i = 1/x_i$, $i = 1, 2, 3$. The four centers of inversion remain fixed under this transformation, and every other point of any cubic of the net is sent into another point on the same curve with reciprocal coördinates. Hence, any pair of these reciprocal points on the Neuberg cubic lies on a line which is parallel to the Euler line.

Such pairs of points are the following:

(*a*) the isodynamic points and the isogonal points respectively,

(*b*) the points A_i' and the three points symmetric to the vertices A_i in the lines $q_{jk}q_{kj}$,

(*c*) the point at infinity on the Euler line and the sixth intersection H of the cubic and the circumcircle,

(*d*) the six vertices of the equilateral triangles constructed on the reference triangle and the six intersections of the lines joining the vertices A_i and the isodynamic points with the cubic,

(*e*) the three intersections of lines on the vertices A_i and parallel to the Euler line with the opposite sides of the reference triangle and the points A_i respectively.

This transformation also sends the circumcircle of the base triangle $(s/x) = 0$ into the ideal line $(sx) = 0$. Again, the diameters of the circumcircle are transformed into equilateral hyperbolas on the vertices of the reference triangle and its orthocenter.[1] In particular the common chord of the circles of Apollonius, $(\sigma x) = 0$, containing the isodynamic centers, is transformed into Kiepert's hyperbola, $(\sigma/x) = 0$, containing the isogonal centers and the point $(1/s)$.

7. An Unsolved Problem. Each one of the three circles of Apollonius of the reference triangle cuts the circumcircle in a second point. These three points form a triangle which is perspective with the base triangle through the symmedian point, the co-symmedian triangle. This triangle also possesses a Neuberg cubic, which has three points in common with the cubic of the first triangle in the circumcenter and the two isodynamic centers. Hence the remaining points of intersection must lie on a conic, which is necessarily a circle, since two of these six points are the circular points I and J. This circle apparently bears an interesting relation to the base triangle, and so far as known, its equation has not yet been found.

[1] Clebsch-Lindemann, *l.c.*, p. 338.

NOTE ON THE PRECEDING PAPER.

By B. H. BROWN, Dartmouth College.

The reader may have noted that some of the properties of the 21-point cubic proved by Messrs. Moore and Neelley were given in my recent article (*1925*, 110–115) on "The twenty-one point cubic." Happily there is no question of priority here, the two papers being independent and essentially simultaneous. The very neat attack in the Moore-Neelley paper is so different from mine that the duplication of results is at once instructive and amusing.

In one respect the papers are similar, namely, in the employment of pencils of cubics. I cannot refrain from commenting on the great utility of these pencils in the study of the elementary geometry of the triangle. A systematic search for sets of 9 notable points through which pass a pencil of cubics would undoubtedly prove profitable. I may be pardoned for mentioning again one such: the vertices, the in- and excenters, the orthocenter, and the circumcenter. The Neuberg cubic is one of this pencil. Another passes through the mid-points of the sides, the centroid, and the symmedian (Lemoine) point.

In this connection my pupil, Mr. L. S. Kennison, announces the following

THEOREM. *The diagonal points of the complete quadrangle whose vertices are the isogonal and the isodynamic centers are the centroid, the symmedian point, and the infinite point on the Euler line.*

The portion of the theorem relative to the point on the Euler line is given in the Moore-Neelley paper and in mine. That the isodynamic centers and the symmedian point are collinear was shown by Neuberg. The rest of this theorem is perhaps new. I have verified Mr. Kennison's proof, a straightforward analytic attack. It would be interesting to have a simple synthetic proof for this very neat theorem.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

I. NOTE ON CERTAIN PRACTICAL PROBLEMS WHICH REQUIRE THE USE OF EXTENDED VALUES OF MATHEMATICAL FUNCTIONS.[1]

BY C. E. VAN ORSTRAND, U. S. Geological Survey.

The increased use of computing machines tends to increase the demand for extended values of both functions and constants. This result is obvious when one consider the ease with which large numbers are handled on a computing

[1] Published with the permission of the Director of the U. S. Geological Survey.

machine. Obviously, there is no need of introducing slight inaccuracies in a final result when the labor involved in making the computation is almost the same as that involved in obtaining a lower degree of accuracy. For example, if one desires to obtain 5-place common logarithms from 5-place natural logarithms, it is preferable to use 6 or even 8 decimals of the modulus, and thus avoid the possibility of a slight accumulation of error in the product.

There is also an increasing demand for large numbers resulting from new developments in science. Consider for instance the problem of determining the number of particles in a cubic centimeter of gas which have velocities falling between certain limits. The total number of particles in a cubic centimeter of gas is approximately 2.705×10^{19}, or approximately 6.062×10^{23} in a gram molecule. The elaborate table of the probability integral by Burgess, a portion of which is tabulated to 15 places of decimals, barely suffices to give the desired information for large values of the argument.

Another very important use of extended values of mathematical functions arises from the demands imposed in the compilation or computation of mathematical tables. Anyone who has attempted this task soon realizes that the number of published papers dealing with extended values of even the most important functions is very limited, and furthermore, the labor of supplying this additional information may be appalling. Numerous interesting developments in series may be made, but such developments are not always easy of evaluation —the series frequently converges too slowly, and the accuracy of the resulting computation is not easily checked. To suggest the difficulty of the task, consider that

$$\sin 0.394 = 0.38388\ 49999\ 93636\ \ldots.$$

The correct tabulation of this value to 5 places of decimals requires the evaluation of the function to at least 12 places of decimals. This instance is somewhat exceptional, but experience proves that the calculations on which mathematical tables are based should be carried to 5 or more additional decimals. The actual error involved may be very small, but a mathematical table should be an absolute standard of reference, otherwise confusion arises in the inter-comparison of tables; and in the numerous practical applications to which the tables are put, small errors are introduced which slightly vitiate the results.

In conclusion, a curious result may be mentioned which arises from the use of computing machines in the evaluation of functions. Heretofore, in the evaluation of a series, the computation proceeded on a term by term basis which generally resulted in the addition of only 2 or 3 digits to the computed value of the function as the successive terms were added. With the computing machine, however, the steps may be increased in favorable cases to 6 or 8 digits, consequently the extra labor involved in obtaining several extra digits may not be of serious importance. It seems highly desirable, in these cases, to tabulate the extra digits; for no one can foresee the exceptional demands which mathematical tables may be required to meet.

II. On the Occasional Need of Very Accurate Logarithms.

By H. Bateman, California Institute of Technology.

Twelve-figure logarithms are sometimes used in computations where great accuracy is needed but, even when it is sufficient to know correctly only a few figures, cases may arise in which twelve- and even twenty-figure logarithms must be used.

To illustrate this, let us suppose that we need the value of the Legendre function [1]

$$Q_n(x) = \tfrac{1}{2}P_n(x)\log\frac{x+1}{x-1} - \sum_{s=1}^{n}\frac{1}{s}P_{n-s}(x)P_{s-1}(x) \quad (x > 1)$$

for values of x that are not far from unity so that a power series in $1/x$ cannot be used for computation.

Except when x is very nearly unity and n is small the value of $Q_n(x)$ is small while that of $P_n(x)$ is large. This is shown by the following figures.[2]

$P_{10}(1.1) = 19.79$, $P_{10}(1.2) = 106.544$, $P_{10}(2) = 96060.52$,
$Q_{10}(1.1) = 0.00528$, $Q_{10}(1.2) = 0.0006756$, $Q_{10}(2) = 0.0000002843$,
$P_{20}(1.1) = 1175.69$, $P_{10}(3) = 8097453$,
$Q_{20}(1.1) = 0.0000453$, $Q_{10}(3) = 0.0000000021$.

The quantity to be calculated is thus the small difference between two large quantities and so the logarithmic factor in the expression for the first of these must be known very accurately. Fortunately J. C. Adams has calculated the logarithms of the first few integers to a large number of decimal places [3] and his results can be used in tabulating functions such as $Q_n(x)$.

Cases analogous to the above may be of frequent occurrence because the solution of a linear differential equation often involves a logarithmic term multiplied by a particular solution which may become large while the solution needed is small and the expression involving the logarithm is the only one available.

There are similar cases of asymptotic expansions or exact representations in which the first term has a trigonometrical factor or a factor containing Gamma functions. Again this first term may be large though the quantity to be calculated is small and so great accuracy is needed. Suitable tables of trigonometrical functions and of $\log\Gamma(1+x)$ have been published in the British Association Reports.[4]

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[1] For this expansion see Whittaker and Watson, *Modern Analysis*, 3d edition, p. 333.

[2] *Messenger of Mathematics*, vol. 52 (1922), p. 71. Spheroidal harmonics are needed for the solution of many potential problems. See for instance *Technical Note 104*, National Advisory Committee for Aeronautics (1922).

[3] *Proc. Roy. Soc. London*, vol. 27 (1878), p. 88.

[4] Year (1916) for 11-figure tables of sines and cosines (J. R. Airey) and 10-figure tables of $\text{Log}_e\,\Gamma(1+x)$ (G. N. Watson). Year (1923) for 15-figure tables of sines and cosines of angles in radians.

III. Solution of the General Biquadratic.

By E. J. Oglesby, New York University.

In a previous number of the Monthly (*1923*, 321) the author gave a solution of the general cubic equation. In the present note the same method is applied to the biquadratic.

Take the equation

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0, \tag{1}$$

and assume as roots

$$\begin{aligned} x_1 &= a + b + c + d, \\ x_2 &= a + b - c - d, \\ x_3 &= a - b - c + d, \\ x_4 &= a - b + c - d, \end{aligned} \tag{2}$$

where a, b, c, and d are to be determined. Using the right-hand members of (2), we have the following identities:

$$x_1 + x_2 + x_3 + x_4 = 4a, \tag{3}$$

$$x_1x_2 + \ldots + x_3x_4 = 6a^2 - 2(b^2 + c^2 + d^2), \tag{4}$$

$$x_1x_2x_3 + \ldots + x_2x_3x_4 = 4a^3 - 4a(b^2 + c^2 + d^2) + 8bcd, \tag{5}$$

$$x_1x_2x_3x_4 = -4(ab - cd)^2 + (a^2 + b^2 - c^2 - d^2)^2; \tag{6}$$

whence

$$4a = -4a_1/a_0, \tag{7}$$

$$6a^2 - 2(b^2 + c^2 + d^2) = 6a_2/a_0, \tag{8}$$

$$4a^3 - 4a(b^2 + c^2 + d^2) + 8bcd = -4a_3/a_0, \tag{9}$$

$$-4(ab - cd)^2 + (a^2 + b^2 - c^2 - d^2)^2 = a_4/a_0. \tag{10}$$

Put

$$\begin{aligned} a_0a_2 - a_1^2 &= H, \\ a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3 &= G, \\ a_0a_4 - 4a_1a_3 + 3a_2^2 &= I. \end{aligned}$$

From (7),

$$a = -a_1/a_0. \tag{11}$$

Substitute for a in (8), and we get

$$b^2 + c^2 + d^2 = -3H/a_0^2. \tag{12}$$

Substitute for a and for $b^2 + c^2 + d^2$ in (9), and we get

$$bcd = -G/2a_0^3. \tag{13}$$

From (12),

$$c^2 + d^2 = -(3H/a_0^2) - b^2. \tag{14}$$

From (13),

$$cd = -G/2a_0^3b. \tag{15}$$

Substitute from (11), (14), and (15) in (10), and we get

$$-4\left(-\frac{a_1b}{a_0}+\frac{G}{2a_0^3b}\right)^2+\left(\frac{a_1^2}{a_0^2}+2b^2+\frac{3H}{a_0^2}\right)^2=\frac{a_4}{a_0},$$
$$-(G-2a_0^2a_1b^2)^2+a_0^2(2a_0^2b^2+3H+a_1^2)^2b^2-a_0^5a_4b^2=0;$$

which reduces to

$$4a_0^6b^6+12Ha_0^4b^4+a_0^2(12H^2-a_0^2I)b^2-G^2=0, \tag{16}$$

our resolvent equation, a cubic in b^2.

The three roots of this equation are b^2, c^2, and d^2. The signs of b, c, and d are determined from (13). Fix any two of the signs and determine the third. Then substitute the values of a, b, c, and d in (2).

Example.

$$x^4-6x^3+12x^2-20x-12=0;$$
$$a_0=1,\quad a_1=-3/2,\quad a_2=2,\quad a_3=-5,\quad a_4=-12;$$
$$H=a_0a_2-a_1^2=-1/4;$$
$$G=a_0^2a_3-3a_0a_1a_2+2a_1^3=-11/4;$$
$$I=a_0a_4-4a_1a_3+3a_2^2=-30.$$

Hence (16) becomes

$$4b^6-3b^4+(123/4)b^2-121/16=0;$$
$$b^2=\tfrac{1}{4},\quad (1+2\sqrt{-30})/4,\quad (1-2\sqrt{-30})/4;$$

and

$$b=\pm\tfrac{1}{2},\quad \pm(\sqrt{6}+i\sqrt{5})/2,\quad \pm(\sqrt{6}-i\sqrt{5})/2.$$

Take

$$c=(\sqrt{6}+i\sqrt{5})/2,\qquad d=(\sqrt{6}-i\sqrt{5})/2.$$

Then, by (13), $bcd=11/8$, so that $b=\frac{1}{2}$; whence

$$x_1=a+b+c+d=2+\sqrt{6},$$
$$x_2=a+b-c-d=2-\sqrt{6},$$
$$x_3=a-b-c+d=1-i\sqrt{5},$$
$$x_4=a-b+c-d=1+i\sqrt{5}.$$

IV. Note on the Introduction of Integral Calculus into a College Course in Solid Geometry.

By J. P. Ballantine, Columbia University.

I believe that anyone who has taught solid geometry in college will agree that there are certain parts which are hardly up to college grade; such problems, for instance, as one in which the area of the base and the altitude of a prism are given, and the student is expected to apply the formula for the volume. Such volumes as the cone, sphere, etc., would be very easily computed by integral calculus, if there were only some easy way to give the student the necessary fundamentals. In the following discussion I propose a possible way in which this may be done.[1]

In the first place, we will need Cavalieri's principle. This may be taken as an unproved principle, or it may be justified by any degree of rigor which the instructor desires up to and including that which would be given in a course on integral calculus.

If h denotes the altitude of a particular solid, then any value of x between 0 and h will determine the cross-section at a distance x from the base. It is conceivable that in the case of a particular solid the area of this cross-section is given by a formula u, such as, for instance, $x^2 - 2x$, which will hold for every value of x between and including 0 and h.

By Cavalieri's principle, two solids whose cross-sections are given by the same formula must have the same volume. This volume is denoted by the symbol

$$\int_0^h u.$$

If, more generally, we are considering only the volume included between the sections $x = a$ and $x = b$, the following symbol $\int_a^b u$ is used.

The following three formulæ are established as consequences of the above two definitions:

$$\int_a^b u = \int_0^b u - \int_0^a u,$$

$$\int_a^b (u + v) = \int_a^b u + \int_a^b v,$$

$$\int_a^b (nu) = n \int_a^b u,$$

where n is any real number.

It is next necessary to construct sample solids with altitudes h and with cross-section formulæ 1, x, and x^2 respectively in order to evaluate the integrals of these three functions. The first is a prism, and its volume is h. The second

[1] Compare the discussion by W. R. Longley (*1924*, 196).

is also a prism whose base is an isosceles right triangle with equal sides h, and whose altitude is 1. The sections which are given by the formula x are taken parallel to one of the faces which is a rectangle h by 1. The volume is $h^2/2$. For the third solid, one starts with a cube $2h$ on an edge. Using the center as a vertex and each face of the cube as a base, six equal square pyramids may be formed, each with one sixth the volume of the cube. The pyramid formed on the upper face of the cube has its cross-section given by the formula $4x^2$. Hence we have the equation

$$\int_0^h 4x^2 = 4\int_0^h x^2 = \tfrac{1}{6}(2h)^3$$

from which the integral of x^2 may be obtained,

$$\int_0^h x^2 = h^3/3.$$

Thus, on the basis of the three formulæ for manipulating integrals, and the values of the three particular integrals, it is possible for the student in solid geometry to integrate any quadratic function. In particular, the volume of any pyramid can be found by this process. The volume of a sphere or of any spherical segment can be obtained by integrating the expression $\pi(r^2 - x^2)$.[1] In place of proving that the volume of every prismatoid is given by the prismatoid formula, one may prove the more interesting and more fundamental theorem that if the cross-section of a solid is given by a quadratic formula, then the volume is given by the prismatoid formula.

V. Two New Arctangent Relations for π.

By A. A. Bennett, University of Texas.

Among the numerous methods used to obtain successive numerical approximations to π, the most familiar is probably that which employs the arctangent function, profiting by the fact that the principal value of arctangent 1 is $\pi/4$. We have for example such familiar relations as the following:

$$\pi/4 = \arctan \tfrac{1}{2} + \arctan \tfrac{1}{3}, \tag{1}$$

$$\pi/4 = 2\arctan \tfrac{1}{3} + \arctan \tfrac{1}{7}, \tag{2}$$

$$\pi/4 = \arctan \tfrac{1}{2} + \arctan \tfrac{1}{5} + \arctan \tfrac{1}{8}, \tag{3}$$

$$\pi/4 = 4\arctan \tfrac{1}{5} + \arctan {}_{2}{}^{1}{}_{3}{}^{1}{}_{9}. \tag{4}$$

Equation (1) gives rise to the series known as Euler's series. Equation (2) was used as a check by Clausen in 1847 in computing π to 248 places of decimals.

[1] The process of carrying out the suggested integration by use of the three formulæ referred to is exactly equivalent to the process, used for instance in Hawkes, Luby, and Touton's *Solid Geometry*, of considering the sphere as the difference between the circumscribing cylinder and a certain two-napped cone.

It had also been used previously by Vega as a check in computing π to 140 places. Equation (3) enabled Dase in 1844 to calculate π to 200 places of decimals. Equation (4) is the justly celebrated relation due to Machin (1706) and forms the basis for numerous computations of π, including the work of the computers mentioned above. In particular, W. Shanks computed π to 707 places by the use of Machin's formula.

All of these relations are suited for use in the power series, called Gregory's series,

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots, \tag{5}$$

which is leisurely convergent even for $x = 1$. Another familiar series for arctangent x is the following:

$$\arctan x = \frac{x}{1+x^2}\left\{1 + \frac{2}{3}\left(\frac{x^2}{1+x^2}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{x^2}{1+x^2}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{x^2}{1+x^2}\right)^3 + \cdots\right\} \tag{6}$$

given by Euler in 1755.

For (6) it is not necessarily most convenient to take x as the reciprocal of an integer. Indeed Euler's relation,

$$\pi/4 = 5 \arctan \tfrac{1}{7} + 2 \arctan \tfrac{3}{79}, \tag{7}$$

is in many ways more convenient for computation than any one of the relations (1), (2), (3), or (4).

The new equations here given are intended for use in Gregory's series. The first is not unsuitable for paper and pencil computations, as it converges much more rapidly than does Machin's formula. The second is selected with particular reference to computing machines. It satisfies the condition that no one of the denominators requires a setting of more than five successive digits on a machine, and moreover most of these denominators are particularly convenient for machine division, since the successive partial quotients are usually immediately obvious. These equations are:

$$\pi/4 = 12 \arctan \tfrac{1}{18} + 3 \arctan \tfrac{1}{70} + 5 \arctan \tfrac{1}{99} + 8 \arctan \tfrac{1}{307}, \tag{8}$$

$$\begin{aligned}\pi/4 = {} & 160 \arctan \tfrac{1}{200} - (\arctan \tfrac{1}{239} + 4 \arctan \tfrac{1}{515} + 8 \arctan \tfrac{1}{4030} \\ & + 16 \arctan \tfrac{1}{80105} + 16 \arctan \tfrac{1}{82575} + 32 \arctan \tfrac{1}{800150} \\ & + 80 \arctan \tfrac{1}{4000800}).\end{aligned} \tag{9}$$

It is easy to verify (8) by means of the sequence of relations,

$$\begin{aligned} &\text{arccot } 5 = \text{arccot } 7 + \text{arccot } 18, && \text{arccot } 7 = \text{arccot } 12 + \text{arccot } 17, \\ &\text{arccot } 12 = \text{arccot } 17 + \text{arccot } 41, && \text{arccot } 17 = \text{arccot } 18 + \text{arccot } 307, \\ &\text{arccot } 41 = \text{arccot } 70 + \text{arccot } 99, && \text{arccot } 70 = \text{arccot } 99 + \text{arccot } 239, \end{aligned}$$

together with (4). Similarly (9) is verified by means of the sequence of relations,

$$\begin{aligned}
\operatorname{arccot} 5 &= 2 \operatorname{arccot} 10 - \operatorname{arccot} 515,\\
\operatorname{arccot} 10 &= 2 \operatorname{arccot} 20 - \operatorname{arccot} 4030,\\
\operatorname{arccot} 20 &= \operatorname{arccot} 25 + \operatorname{arccot} 100 - \operatorname{arccot} 50105,\\
\operatorname{arccot} 25 &= 2 \operatorname{arccot} 50 - \operatorname{arccot} 62575,\\
\operatorname{arccot} 50 &= 2 \operatorname{arccot} 100 - \operatorname{arccot} 500150,\\
\operatorname{arccot} 100 &= 2 \operatorname{arccot} 200 - \operatorname{arccot} 4000300,
\end{aligned}$$

together with (4).

The extraordinarily rapid convergence of the several Gregory series arising from (9) makes this formula particularly useful unless only very few significant figures of π are desired.

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

La "Thiende" de Simon Stevin. Fac-simile edition with an introduction by H. BOSMANS, S.J. Antwerp, Société des Bibliophiles Anversois, 1924. vi + 37 pages. Price 15 Belgian francs.

All who have interest in the epoch-making classics of mathematics will be glad to know of this recent reproduction in fac-simile of the first treatise ever published upon decimal fractions. The original work was issued in the Flemish language at Leyden, from the local branch of the Plantin Press, in 1585. It is now excessively rare, and it was only by good fortune that Père Bosmans happened to have the Louvain copy at his home when the library was burned during the World War. The French translation made in the same year, and also published by Plantin, is better known, particularly in the edition of *L'Arithmétique*.

Naturally it will be the French edition that will be generally consulted by students, unless they know of Miss Sanford's excellent translation which appeared in the *Mathematics Teacher* in October, 1921. It will, however, be of interest to many readers to have a fac-simile of the original edition. Thanks to the generous action of the Société des Bibliophiles Anversois (No. 22, Marché du Vendredi, Antwerp), it is possible to obtain a copy at a merely nominal price.

All students of the history of mathematics are indebted to Père Bosmans for his many valuable essays, and it will be of interest to readers to know that he has prefaced this edition with an introduction of much value to bibliophiles as well as to those who make a study of the origins of our science.

D. E. SMITH.

Fundamental Topics in the Differential and Integral Calculus. By GEORGE RUTLEDGE. New York, Ginn and Company, 1923. 252 pages. Price $2.00.

This book, which is intended for use in a liberal arts college where only one semester of mathematics is required, is not a unified course but simply an elementary course in the Calculus. The topics presented are variables; functions; graphs; continuity of a function and of its graph; derivative of a polynomial; slope of a graph; the sine function and its derivative; the square-root function and its derivative; differentials; the differential as an approximation to the increment; rates, velocity; acceleration; derivative of a product or quotient; derivative of a function of a function; differential of length; properties of linear, quadratic and cubic polynomial graphs; the square root of a quadratic polynomial; maxima and minima; definite integrals; area; length; volume; integration process in general; summation by substitution; the logarithmic function; the exponential function; the hyperbolic functions; inverse circular and hyperbolic functions; derivative equations; differential equations; what is calculus?

Thus if one believes in scrapping most of trigonometry, college algebra and analytic geometry and giving arts students in the minimum amount of time an insight into the Calculus, he will find this an admirable book. The book contains some excellent graphic illustrations and an abundance of well-graded exercises.

F. M. MORGAN.

Mathematical Theory of Life Insurance. By C. H. FORSYTH. New York, John Wiley and Sons, 1924. 74 pages. Price $1.25.

In this little book of 74 pages are presented the rudiments of life insurance in a delightful style. There is very little that is new; but what is contained in the text is brief, necessary, and well put.

Beginning with a short review of probabilities and interest, the author then gives the mortality table and expectations of life. Annuities are introduced by a study of the pure endowment ${}_nE_x$, and assurances are treated by means of the single net premium ${}_nI_x$; and in the chapter on commutation symbols the American N_x is used. There is a good chapter on valuation of policies, and finally one on the laws of Gompertz and Makeham.

A student who has mastered this little text will be well prepared for further work in the mathematics of life insurance.

C. T. BUMER.

Mathematical Analysis of Statistics. By C. H. FORSYTH. New York, John Wiley and Sons, 1924. 241 pages. Price $2.25.

There is at present a considerable interest on the part of many mathematicians in the problems of statistics, and any new book on the subject is sure to attract attention. At the present stage of the development the time is probably not ripe for putting the material of mathematical statistics in final form: and in the meantime, books such as this one satisfy a real need.

The book opens with an introductory chapter on errors and numerical computation, followed by two chapters on finite differences and applications to interpolation. Just why the Gamma function is introduced is not clear. The chapter is too difficult for the student who has not had a course in the calculus, and rather too brief for one who has had such a course. The work on probability, averages, dispersion, etc., covers the usual material. The method of moments has been applied to curve fitting and to the analysis of time series. In analyzing the time series the author uses the method originated by W. M. Persons for eliminating seasonal variation. Various types of statistical series are discussed after the method of Arne Fisher. The treatment of the normal curve, least squares, and probable error make up one chapter; and finally there is a short, sketchy lesson on the theory of correlation.

There is a certain abruptness in the book in places; but the impression arrived at, after using it in the classroom, is that the text is alive—something which cannot be said of most books on statistics. There are many problems throughout, some new and some old, which aid considerably in fixing in mind the theory explained.

C. T. BUMER.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 30, nos. 9–10, November–December, 1924: "On simple groups of low order" by F. N. Cole, 489–492; "Complete class number expansions for certain elliptic theta constants of the third degree" by E. T. Bell, 493–496; "Note on a special congruence" by M. C. Foster, 496–503; "Concerning relatively uniform convergence" by R. L. Moore, 504; "The theory of closure of Tchebycheff polynomials for an infinite interval" by J. A. Shohat, 505–510; "Nuclear and hypernuclear points in the theory of abstract sets" by E. W. Chittenden, 511–519; "Five axioms for point and translation in affine geometry" by A. A. Bennett, 520–526; "Some problems of closure connected with the Geiser transformation" by A. Emch, 527–535; "A symmetric coefficient of correlation for several variables" by D. Jackson, 536–542; "Methods for finding factors of large integers" by H. S. Vandiver, 542–553.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 11, no. 1, January, 1925: "Notes on stellar statistics: The mathematical expression of the law of tangential velocities" by W. J. Luyten, 87–90; "Economics and the calculus of variations" by G. C. Evans, 90–95; "Separable quadratic differential forms and Einstein solutions" by E. Kasner, 95–96; "On a new method of factorization" by D. N. Lehmer, 97–98; "Functionals of curves admitting one-parameter groups of infinitesimal point transformations" by A. D. Michal, 98–101.

SCHOOL SCIENCE AND MATHEMATICS, volume 25, no. 2, February, 1925: "Hurdle tests in algebra" by K. Wentz, 132–144; "Arithmetic in the junior high school" by L. W. Colwell, 171–178.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB TOPICS.

1925 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS.

By W. C. EELLS, Whitman College, Walla Walla, Wash.

Program committees of college and high school mathematics clubs, teachers of courses in the history of mathematics, and others may be interested in having called to their attention important events in the history of mathematics for which 1925 is the centennial year.

Many of these events suggest timely topics for club programs in the form of centennial commemorations of occurrences which took place an even number of centuries preceding the present year.

Ample material in English for the development of most of the suggested topics may be found in Ball's *Short History;* Smith's *History;* Cajori's *History* (Revised), *Elementary History,* and *History of Mathematics in the United States;* Fink's *Brief History;* Heath's *History of Greek Mathematics,* and special treatises on *Euclid* and *Archimedes;* Allman's *Greek Geometry from Thales to Euclid;* and in general encyclopedias and other works of reference.

Below is given a list of such centennial dates, accompanied in some cases by brief suggestions of closely related topics for investigation.

375 B.C. Approximate date of birth of Menæchmus. Conic sections.

375 B.C. Theætetus flourished. Greek theory of irrationals; five regular solids.

275 B.C. Approximate date of death of Euclid, who for twenty-two centuries through the *Elements* with its more than two thousand editions "has exercised such profound influence on the teaching and knowledge of geometry."

275 B.C. Approximate date of birth of Eratosthenes. Construction of tables of prime numbers; modern factor tables; earliest measurement of arc of earth's surface.

75 B.C. Discovery by Cicero of tomb of Archimedes, "the great geometer," "the greatest mathematician of antiquity." Geometry of the sphere and cylinder.

125 A.D. First astronomical observations of Ptolemy. The Almagest; Greek Astronomy.

1025. Approximate date of Sridhara's *Trisatika.* Early Hindu mathematics.

1225. Publication of Leonardo of Pisa's *Liber Quadratorum.* (MONTHLY, *1919,* 1–8.)

1225. First "Mathematical Tournament." Other mathematical tournaments of history.

1225. Birth of Thomas Aquinas. Infinity and the linear continuum.
1525. Publication of Rudolff's *Die Coss*, first German text-book of algebra. Introduction of symbols for square root.
1525. Publication of Dürer's *Underweysung der Messung*, with derivation of the Epicycloid. History of the Epicycloid.
1625. (Mar. 5) Birth of Collins. Newton-Leibnitz controversy.
1625. (June 8) Birth of Cassini. Arc of earth's surface; the Cassinian Oval.
1625. Birth of de Witt. Projective generation of conics.
1725. Clairaut, at age of 12, writes important paper on "Four Geometric Curves." Other mathematical prodigies.
1725. (Sept. 5) Birth of Montucla. Historians and histories of mathematics.
1825. (Mar. 10) Death of Mollweide. "Gauss' Analogies."
1825. (Apr. 21) Death of Pfaff. Differential equations; "Pfaffian problem."
1825. Bolyai's development of the theory of parallels. Non-Euclidean Geometry.
1825. Publication of final volume of Laplace's "*Mécanique Céleste*," "the translation of the *Principia* into the language of the differential calculus" (Ball), "a complete solution of the great mechanical problem presented by the solar system."
1825. Publication of first volume of Legendre's monumental treatise *Fonctions Elliptiques*.
1825. Publication in United States of Warren Colburn's *Algebra*. Colburn's other works; early American algebra.
1825. Founding of the *Mathematical Diary*, by Robert Adrain. Other early American mathematical journals.
1825. Benjamin Peirce, "father of American mathematics," enters Harvard University (MONTHLY, *1925*, p. 9).

CLUB ACTIVITIES.

THE MATHEMATICAL CLUB OF SMITH COLLEGE, Northampton, Massachusetts. [*1922*, 26.]

The officers for the year 1923–1924 were: President: Charlotte Nelson, '24; Secretary: Emily Wilson, '24; Treasurer: Dorothy Williams, '24.

The programs for the past two years are as follows:

October 22, 1923: "Extracts from the history of mathematics"—Charlotte Nelson, '24. Mathematical puzzles.

November 12, 1923: "Mathematical curricula in foreign countries"—Carolyn Waterbury, '24. Puzzles.

December 3, 1923: Christmas party at the home of Professor Wood. Games, puzzles, grinds, and presents.

February 18, 1924: "Demonstration of the Chinese method of calculating with the abacus"—Professor Cobb.

March 10, 1924: "Quadric surfaces"—Evelyn Winters, graduate student.

April 14, 1924: "Teaching mathematics"—Dorothy Lane, '25. Elizabeth Lane, '25, showed a hyperbolic paraboloid which she had made.

May 5, 1924: Reading from "Flatland"—Charlotte Nelson. "Hindu numerals"—Professor Benedict.

May 29, 1924: Farewell party of the year. A stunt was given showing the humorous side of our mathematics classes and teachers.

Officers for the year 1924–1925 are: President: Elizabeth Parkhurst, '25; Secretary: Mary Mangan, '25; Treasurer: Elizabeth Lane, '25.

October 20, 1924. Plans for the year were discussed. Members of the club volunteered to give papers through the year.

November 10, 1924: "History of mathematical education"—Hope Adams, '25.

December 1, 1924: Christmas party at Professor Wood's. Games were played and a stunt was given.

January 12, 1925: "Babylonian and Egyptian mathematics"—Doris Latimer, '25. "John Napier"—Rachel Lorrey, '26.

(Report by Miss Mangan, Secretary.)

Pi Mu Epsilon, University of Illinois, Urbana, Illinois.

The Illinois chapter of Pi Mu Epsilon is an organization which has for its aims, first, scholarship for individual members in all subjects and particularly in mathematics; second, the advancement of the science of mathematics; and last, the common and personal advancement of its members.

It is the custom of our chapter to initiate new members once each semester, the initiation for the first semester of the year 1924–1925 having taken place on December tenth, at which time seventeen new members were initiated into the organization, three members of the mathematics faculty, one graduate student, and thirteen others including engineers and mathematics majors. At the present time Illinois chapter has forty-six members, a body made up of juniors, seniors, and graduate students whose work has shown unusual interest and ability in mathematics or in the mathematical sciences, and members of the mathematics faculty among whom is Dr. E. J. Townsend, head of the mathematics department.

In the pursuance of its aims the chapter holds fortnightly meetings open to the public at each of which an address of some mathematical interest is given by some member of the organization. The open meetings of the first semester of the year 1924–1925 were as follows:

October 15: "Sturm's theorem," Dr. J. B. Shaw.

November 5: "Mathematical games," Dr. A. J. Kempner.

December 3: "The nine point circle," H. L. Black.

January 14: Report of National Convention of Pi Mu Epsilon, Dr. R. D. Carmichael.

The good attendance at these meetings shows that they have been of considerable interest both among faculty members and students.

(Report by Miss Waters, Secretary.)

The Irrational Club, University of Wyoming, Laramie, Wyoming.

The Irrational Club of the University of Wyoming, Laramie, Wyoming, started this year. Its membership is open to all who are interested in mathematics. The officers of the club are as follows:

Positive Square Root.....Robert Burns.
Negative Square Root.....Oswald Seaverson.
Keeper of the Log and Bones.....Marcella Avery.
Custodian of the Indices.....The above fundamental elements of the club with the addition of Miss Gretna Neubauer and Clark Beisemeier.

The following subjects have been or will be discussed during the coming year: Magic squares, Flatland, Women and mathematics, Paper folding, History of mathematical symbols, History of logarithms and how to use them, Map drawing, Mathematics of astronomy, Fourth dimension.

A portion of each meeting is spent in a five-minute history sketch; a like amount of time is devoted to some mathematical recreation and some phase of computation; thirty minutes is given to the major topic. In addition to these items place is provided on the program for the solution of problems.

(Report by Professor H. C. Gossard.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. A. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3133. Proposed by A. A. BENNETT, University of Texas.

Show that in every set containing but a finite number of elements and admitting an associative rule of multiplication, there must be an idempotent element, that is, one which is equal to its square. Show that this does not continue to hold if the set contains an infinite number of elements.

3134. Proposed by J. ROSENBAUM, Milford, Connecticut.

In a given circle to inscribe a hexagon of assigned angles which shall have the maximum area.

3135. Proposed by N. P. PANDYA, Amreli, Kathiawad, India.

Form the equation of lowest degree with rational coefficients whose roots are $\sin A$, $\sin 4A$, $\sin 7A$, and $\sin 10A$, where $13A = \pi$.

3136. Proposed by PAUL CAPRON, U. S. Naval Academy.

If the sines of the half-sides of a spherical triangle are h, k, l, and the cosines of the halves of the corresponding opposite angles are H, K, L, and if p, P are the polar distances of the inscribed and circumscribed circles,

$$\sin P = \frac{hkl}{2\sqrt{\sigma(\sigma-h)(\sigma-k)(\sigma-l)}}; \qquad \cos p = \frac{HKL}{2\sqrt{\Sigma(\Sigma-H)(\Sigma-K)(\Sigma-L)}},$$

where $\sigma = (h+k+l)/2$ and $\Sigma = (H+K+L)/2$.

3137. Proposed by HARRY LANGMAN, New York City.

Show how to draw, using straight-edge only, a tangent to the circumference (or an arc) of a circle at a given point, without making use of Pascal's hexagon theorem.

3138. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

The vertex of a triangle, whose base is fixed, moves along a straight line coplanar with the line of the base. Find (1) the locus of the orthocenter of the triangle; (2) the envelope of the line joining the feet of the two altitudes dropped from the two fixed vertices of the triangle.

3139. Proposed by C. N. MILLS, South Dakota State Normal School.

A rope weighs w pounds per foot. How many coils of the rope must be taken around a rough cylinder of radius r in order that a weight may support another n times as great, k being the coefficient of friction between the rope and the cylinder.

SOLUTIONS.

2850 [1920, 377]. Proposed by SARAH BEALL, U. S. Coast and Geodetic Survey.

An unknown star is observed at the altitudes h_1 and h_2 at the respective times t_1 and t_2, the latitude being known also. Obtain formulas for the right ascension and declination of the star: (1) when the time interval $t_1 - t_2$ is large; (2) when the time interval is so small that $(h_2 - h_1)/(t_2 - t_1)$ may be taken as the value of dh/dt corresponding to the mean altitude $(h_1 + h_2)/2$, and the mean time $(t_1 + t_2)/2$. This problem sometimes arises when a bright star is observed through the clouds.

SOLUTION BY C. C. WYLIE, University of Illinois.

Notation: Let z = zenith distance = $90° - h$; t = sidereal time; D = declination; A = right ascension = $t - P$; L = latitude; and P = hour angle.

Case 2. From one of the fundamental formulæ of spherical trigonometry we have

$$\cos z = \sin D \sin L + \cos D \cos L \cos P. \tag{1}$$

Differentiating,

$$\sin z\, dz = \cos D \cos L \sin P\, dP;$$

whence

$$\sin P = \frac{\sin z}{\cos D \cos L}\frac{dz}{dP}. \tag{2}$$

Equations (1) and (2) can be considered two equations in the two unknowns D and P. We eliminate P by obtaining $\cos P$ from (2).

$$\cos P = \frac{\sqrt{\cos^2 D \cos^2 L - \sin^2 z \left(\frac{dz}{dP}\right)^2}}{\cos D \cos L}. \tag{3}$$

Substituting in (1), and squaring, we have

$$\cos^2 z - 2 \sin D \sin L \cos z + \sin^2 D \sin^2 L = \cos^2 L(1 - \sin^2 D) - \sin^2 z \left(\frac{dz}{dP}\right)^2. \tag{4}$$

Rearranging,

$$(\sin^2 L + \cos^2 L) \sin^2 D + (-2 \sin L \cos z) \sin D + \left\{\cos^2 z + \sin^2 z \left(\frac{dz}{dP}\right)^2 - \cos^2 L\right\} = 0. \tag{5}$$

Denoting the coefficient of $\sin^2 D$ by a, that of $\sin D$ by b, and the other term by c, equation (5) becomes, since $a = 1$,

$$\sin^2 D + b \sin D + c = 0; \tag{6}$$

whence

$$D = \sin^{-1}\left\{\frac{-b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}\right\}. \tag{7}$$

Having D, $\cos D$ can be taken from tables or computed from $\sin D$, and we have from (2)

$$P = \sin^{-1}\left\{\frac{\sin z}{\cos D \cos L}\frac{dz}{dP}\right\},$$

and finally

$$A = t - P \tag{8}$$

which completes the solution of case (2).

Case 1. For case (1) we have $P_2 - P_1 = t_2 - t_1$; and the fundamental equation becomes

$$\begin{cases}\cos z_1 = \sin D \sin L + \cos D \cos L \cos P_1, \\ \cos z_2 = \sin D \sin L + \cos D \cos L \cos P_2.\end{cases} \tag{9}$$

Adding and subtracting we have

$$\begin{cases}\cos z_2 + \cos z_1 = 2 \sin D \sin L + \cos D \cos L (\cos P_2 + \cos P_1), \\ \cos z_2 - \cos z_1 = \cos D \cos L (\cos P_2 - \cos P_1).\end{cases} \tag{10}$$

From the well-known formulæ for the sum and difference of cosines we obtain

$$\cos \tfrac{1}{2}(z_2 - z_1) \cos \tfrac{1}{2}(z_2 + z_1) - \sin D \sin L = \cos D \cos L \cos \tfrac{1}{2}(P_2 - P_1) \cos \tfrac{1}{2}(P_2 + P_1); \tag{11}$$

$$\sin \tfrac{1}{2}(z_2 - z_1) \sin \tfrac{1}{2}(z_2 + z_1) = \cos D \cos L \sin \tfrac{1}{2}(P_2 - P_1) \sin \tfrac{1}{2}(P_2 + P_1). \tag{12}$$

From (12),

$$\sin \tfrac{1}{2}(P_2 + P_1) = \frac{\sin \tfrac{1}{2}(z_2 - z_1) \sin \tfrac{1}{2}(z_2 + z_1)}{\cos D \cos L \sin \tfrac{1}{2}(P_2 - P_1)}. \tag{13}$$

Whence,

$$\cos \tfrac{1}{2}(P_2 + P_1) = \frac{\sqrt{\cos^2 D \cos^2 L \sin^2 \tfrac{1}{2}(P_2 - P_1) - \sin^2 \tfrac{1}{2}(z_2 - z_1) \sin^2 \tfrac{1}{2}(z_2 + z_1)}}{\cos D \cos L \sin \tfrac{1}{2}(P_2 - P_1)}. \quad (14)$$

Substituting this value of $\cos \frac{1}{2}(P_2 + P_1)$ in (11) and squaring we have

$$\begin{aligned} &\cos^2 \tfrac{1}{2}(z_2 - z_1) \cos^2 \tfrac{1}{2}(z_2 + z_1) - 2 \sin D \sin L \cos \tfrac{1}{2}(z_2 - z_1) \cos \tfrac{1}{2}(z_2 + z_1) + \sin^2 D \sin^2 L \\ &\quad = \cos^2 L(1 - \sin^2 D) \cos^2 \tfrac{1}{2}(P_2 - P_1) - \sin^2 \tfrac{1}{2}(z_2 - z_1) \sin^2 \tfrac{1}{2}(z_2 + z_1) \cot^2 \tfrac{1}{2}(P_2 - P_1). \end{aligned} \quad (15)$$

Rearranging,

$$\begin{aligned} &\{\sin^2 L + \cos^2 L \cos^2 \tfrac{1}{2}(P_2 - P_1)\} \sin^2 D + \{-2 \sin L \cos \tfrac{1}{2}(z_2 - z_1) \cos \tfrac{1}{2}(z_2 + z_1)\} \sin D \\ &\quad + \cos^2 \tfrac{1}{2}(z_2 - z_1) \cos^2 \tfrac{1}{2}(z_2 + z_1) + \sin^2 \tfrac{1}{2}(z_2 - z_1) \sin^2 \tfrac{1}{2}(z_2 + z_1) \cot^2 \tfrac{1}{2}(P_2 + P_1) \\ &\quad - \cos^2 L \cos^2 \tfrac{1}{2}(P_2 - P_1) = 0. \end{aligned} \quad (16)$$

Using the same notation as in the preceding case, we can write (16) in the form

$$a' \sin^2 D + b' \sin D + c' = 0.$$

Whence,

$$D = \sin^{-1}\left\{\frac{-b'}{2a'} \pm \frac{\sqrt{b'^2 - 4a'c'}}{2a'}\right\}. \quad (17)$$

With this value of D, obtain $\cos D$, and from (13) we have

$$\tfrac{1}{2}(P_2 + P_1) = \sin^{-1}\left\{\frac{\sin \tfrac{1}{2}(z_2 - z_1) \sin \tfrac{1}{2}(z_2 + z_1)}{\cos D \cos L \sin \tfrac{1}{2}(P_2 - P_1)}\right\}.$$

The remainder of the solution is

$$P_2 = \tfrac{1}{2}(P_2 + P_1) + \tfrac{1}{2}(P_2 - P_1) \quad \text{or} \quad P_1 = \tfrac{1}{2}(P_2 + P_1) - \tfrac{1}{2}(P_2 - P_1),$$
$$A = t_2 - P_2, \qquad A = t_1 - P_1.$$

2884 [1921, 139]. Proposed by E. H. MOORE, University of Chicago.

Consider an $m \times n$ array A of numbers a_{st} and an $n \times m$ array B of numbers b_{tu}. Show that the system of mn equations:

$$\sum_{tu} a_{st} b_{tu} a_{uv} = 0 \quad (sv),$$

implies the equation:

$$\sum_{tu} a_{st} b_{tu} = 0.$$

The suffixes s, u have the range $1, 2, \cdots, m$ and the suffixes t, v have the range $1, 2, \cdots, n$.

SOLUTION BY H. E. BRAY, The Rice Institute.

Let the rank of the matrix A be $r \leq m$. For simplicity, suppose that the first r rows of A are linearly independent—this situation can be brought about by shifting rows of A and at the same time shifting the corresponding columns of B without affecting the hypothesis or the conclusion.

It follows then, by the theory of linear equations, that r vectors

$$c_{l,g}, \qquad l = 1, 2, \cdots n; \; g = 1, 2, \cdots r,$$

can be determined in such a way that

$$\sum_{l=1}^{n} a_{il} c_{lg} = \delta_{i,g}, \qquad i, g = 1, 2, \cdots r; \quad (1)$$

where $\delta_{ig} = 1$ or 0 according as i is or is not equal to g. For the matrix of the coefficients of $c_{1g}, c_{2g}, \cdots c_{ng}$ contains at least one non-vanishing r-rowed determinant. And this is true for each g.

Again, since A is of rank r, every vector in the last $m - r$ rows of A can be expressed as a linear combination of the first r vectors. Hence every one of the m vectors of A can be expressed as a linear combination of the first r; *i.e.*, constants λ_{kp} can be determined so that

$$\sum_{p=1}^{r} \lambda_{kp} a_{pl} = a_{kl}, \qquad k = 1, 2, \cdots m; \; l = 1, 2, \cdots n. \tag{2}$$

By hypothesis,

$$\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{n} a_{ij} b_{jk} a_{kl} c_{li} = 0.$$

But

$$\begin{aligned}
\sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{n} a_{ij} b_{jk} a_{kl} c_{li} &= \sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{p=1}^{r}\sum_{i=1}^{r}\sum_{l=1}^{n} b_{jk}\lambda_{kp} a_{pl} c_{li} a_{ij} && \text{by (2),} \\
&= \sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{p=1}^{r}\sum_{i=1}^{r} b_{jk}\lambda_{kp}\delta_{pi} a_{ij} && \text{by (1),} \\
&= \sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{p=1}^{r} b_{jk}\lambda_{kp} a_{pj} \\
&= \sum_{j=1}^{n}\sum_{k=1}^{m} b_{jk} a_{kj} && \text{by (2).}
\end{aligned}$$

Hence $\sum_{j=1}^{n}\sum_{k=1}^{m} b_{jk} a_{kj} = 0.$

2989 [1922, 356]. Proposed by L. M. HOSKINS, Stanford University.

How should the following questions be answered, assuming that the place referred to is in latitude 34° 8′?

A building twelve feet high has been erected 49 inches south of our lot line. We desire to erect a wall on our line six inches in thickness. (*a*) How high can we build the wall and have it wholly within the shadow cast by the building? (*b*) How high can we build the wall and have it within the shadow cast by the building during the winter months?

SOLUTION BY C. C. WYLIE, University of Illinois.

Case (*a*): As the problem is stated the answer is obviously "impossible." The latter part of June the sun shines on the north side of the building for some six hours each day and as the shadow is then to the south, a wall to the north could not be "wholly within the shadow cast by the building."

Case (*b*): Strictly speaking, the answer to this is also "impossible." Winter lasts until the sun reaches the Vernal Equinox, at which time it rises due east and nothing to the north of the building could be within its shadow at the time of sunrise.

The problem might be restated in several ways, but if we are concerned with any other time than local apparent noon, it is necessary to know something about the length and position of the wall as compared with the building. The solution for local apparent noon is quite simple and is as follows:

Case (*a*): The height in inches may be

$$144 - 55 \cot (34° 8' - 23° 30') = 144 - 293 = -149 \text{ inches.}$$

This means that if the wall were built 0 inches high the building would have to be 12 ft. 5 in. higher to have the wall in its shadow at the time of noon. We have seen before that the sun shines on the north side of the building for more than six hours each day about June 21, and it is now seen that the wall would be very much in the sunshine at the time of noon then.

Case (*b*): The height of the wall for this case may be

$$144 - 55 \cot (34° 8') = 144 - 81 = 63 \text{ inches.}$$

From this we see that the wall may be built about 5 ft. 3 in. in height and be in the shadow of the building at the time of local apparent noon, throughout the months of fall and winter.

Refraction, semidiameter, annual variation of sun's position, etc., have been neglected in the solution.

3059 [1924, 101, 502]. Proposed by DANIEL KRETH, Wellman, Iowa.

Given the perimeter and the radius of the inscribed and circumscribed circles, to construct the triangle and calculate the lengths of its sides.

NOTE BY J. ROSENBAUM, Milford, Conn., AND OTTO DUNKEL, Washington University.

This note disposes of the question in the Remark (*1924*, 502) regarding the case $R = 2r$ and completes the conditions to be placed upon s, r and R for the existence of a real solution of the problem. It is assumed that s, r and R are real and greater than zero. If the roots of the x equation, given in the solution, are all real, they must be greater than zero; this must also be true of the y equation given in the Remark. It then follows that the three roots of the first equation must be the sides of a triangle, since $s - a$, $s - b$, $s - c$ are each greater than zero.

Suppose then that r and R are two given positive quantities and that the positive quantity s is to be determined so that the roots of either equation are all real. The necessary and sufficient condition for real roots is

$$s^2(9r^2 + s^2 - 18rR)^2 - (s^2 - 3r^2 - 12rR)^3 \leqq 0. \tag{1}$$

Set

$$\begin{aligned} s_1^2 &= 2R^2 + 10rR - r^2 + 2\sqrt{R(R-2r)^3}, \\ s_2^2 &= 2R^2 + 10rR - r^2 - 2\sqrt{R(R-2r)^3}. \end{aligned} \tag{2}$$

Then (1) is true when and only when

$$s_2 \leqq s \leqq s_1, \qquad R \geqq 2r. \tag{3}$$

These results may also be obtained geometrically. If the two circles have a given position, it is known from geometry that if there exists one triangle inscribed in the R circle and circumscribing the r circle, then there exists an infinite number of such triangles including two isosceles triangles. It is easily shown by geometry that the necessary and sufficient condition for an isosceles triangle is

$$d^2 = R^2 - 2rR, \tag{4}$$

where d is the distance between the centers. This must therefore be the necessary and sufficient condition for the existence of any such triangle. If we examine the figure for the isosceles triangle, we obtain by aid of this relation the values of the equal sides,

$$\begin{aligned} x_1^2 &= 2R[R + r + \sqrt{R(R-2r)}], \\ x_2^2 &= 2R[R + r - \sqrt{R(R-2r)}], \end{aligned} \tag{5}$$

and also the corresponding values of s given in (1).

Considering now the cubic curve defined by the x equation, we see that if there is a value of s giving a real solution then a straight line parallel to the x axis at the distance s above cuts the cubic in three real points. The results (5) give the conditions that two values of x coincide in x_1 and x_2. Hence the derivative of s with respect to x is zero at these points. Moreover the x equation has three real and positive roots when and only when s lies in the interval defined by the corresponding s values (3), as may be seen from the form of the curve.

We have then three cases,

I. $R < 2r$. No real triangle.

II. $R = 2r$. Here $s = s_1 = s_2 = 3\sqrt{3}\, r$, and there are three equal roots $a = b = c = 2\sqrt{3}\, r$.

III. $R > 2r$. Here s may have any value within the limits set above and for each such value there exists a real triangle (also the congruent and symmetrically placed triangle). If s is equal to either extreme value, the triangle is isosceles.

If $2 < R/r < 1 + \sqrt{2}$, there is no right triangle; if $R/r = 1 + \sqrt{2}$, there is a single isosceles right triangle; if R/r is greater than this last number, there is a pair of congruent right triangles.

We have then the following theorem: If r and R are any two positive numbers such that $R > 2r$, and if two circles be drawn with these two numbers as radii, and with their centers at the distance $\sqrt{R(R-2r)}$, there are an infinite number of triangles inscribed in the R circle

and circumscribing the r circle. The triangle with the maximum perimeter is isosceles with the perimeter $2s_1$, while the one with minimum perimeter is also isosceles with the perimeter $2s_2$, where the quantities s_1 and s_2 as well as the lengths of the equal sides x_1 and x_2 are defined above. See **3070** [*1924*, 206]. If $R = 2r$, there are an infinite number of triangles, but they are all congruent equilateral triangles.

3091 [1924, 353]. Proposed by PHILIP FITCH, Denver, Colorado.

The top of a grain hopper is a square whose side is 10 feet. The sides of the hopper are portions of right circular cylindrical surfaces, 200 feet in diameter. If the cylindrical surfaces meet the plane of the top at right angles and if the hole in the bottom of the hopper is one foot square, what is the volume of the hopper and how many bushels of grain will be taken out by lowering the level of the grain one foot?

SOLUTION BY G. A. LYLE, Lehigh University.

Take for the cylindrical surfaces which form the sides of the hopper (1) $x^2 + (y+5)^2 = 100$; (2) $x^2 + (z+5)^2 = 100$; (3) $x^2 + (y-5)^2 = 100$; (4) $x^2 + (z-5)^2 = 100$. The planes of the square top and of the square hole in the bottom are then $x = 0$ and $x = \sqrt{279}/2$ respectively. The required volume is four times the volume in the first octant enclosed by the coördinate planes, the cylindrical surfaces (1) and (2) and the planes of the top and bottom.

$$4\int_0^{\sqrt{279}/2} y^2 dx = 4\int_0^{\sqrt{279}/2} (125 - x^2 - 10\sqrt{100 - x^2})dx$$

$$= (297\sqrt{297}/2) - 2000 \sin^{-1}(\sqrt{297}/20) = 503.6 \text{ cu. ft.}$$

Amount of grain taken out when the level of the grain is lowered one foot is

$$4\int_0^1 (125 - x^2 - 10\sqrt{100 - x^2})dx = 99.3 \text{ cu. ft.} = 79.8 \text{ bushels.}$$

3092 [1924, 353]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

What must be the relations between the coefficients of a cubic equation in order that its roots considered as lengths, shall form a triangle?

SOLUTION BY NINA MAY ALDERTON, Mills College, California.

Suppose a, b, c to be the real positive roots of the equation

$$x^3 - px^2 + qx - f = 0.$$

Then p, q and f are all positive since $a + b + c = p$; $ab + ac + bc = q$ and $abc = f$. Now the area S of the triangle whose sides are a, b and c is

$$S = \sqrt{\frac{p}{2}\left(\frac{p}{2} - a\right)\left(\frac{p}{2} - b\right)\left(\frac{p}{2} - c\right)} = \tfrac{1}{4}\sqrt{p(p-2a)(p-2b)(p-2c)}$$

$$= \tfrac{1}{4}\sqrt{p[p^3 - 2(a+b+c)p^2 + 4(ab+ac+bc)p - 8abc]}$$

$$= \tfrac{1}{4}\sqrt{p[4pq - p^3 - 8f]}.$$

Now since p is positive, the condition that S be a real positive quantity is

$$4pq > p^3 + 8f.$$

This condition together with the condition that p, q and f are all positive are the conditions that the roots of the cubic $x^3 - px^2 + qx - f = 0$ may represent the lengths of the sides of a triangle.

NOTE ON THE ABOVE SOLUTION BY H. L. OLSON.

In order that a, b and c be real it is necessary and sufficient that the discriminant,

$$D = 18pqf - 4p^3f + p^2q^2 - 4q^3 - 27f^2,$$

of the cubic be positive. Then if p, q and f are positive, the roots a, b and c will, by Descartes' rule of signs, be all positive.

It remains to be proved that each factor under the radical sign is positive. p, of course, is positive, and if $p - 2a$ and $p - 2b$, for example, were negative, their sum, $2c$, would be negative, contrary to previous proof.

Thus the necessary and sufficient condition that a, b and c form a triangle is that p, q, f,

$$18pqf - 4p^3f + p^2q^2 - 4q^3 - 27f^2,$$

and

$$4pq - p^3 - 8f$$

be positive.

3093 [1924, 353]. Proposed by FRANK MORLEY, Johns Hopkins University.

Show that the equation

$$y = (a_0 + a_1x + \cdots + a_nx^n)/(b_0 + b_1x + \cdots + b_nx^n)$$

gives the differential equation

$$\begin{vmatrix} y_1, & y_2, & \cdots & y_{n+1} \\ y_2, & y_3, & \cdots & y_{n+2} \\ \cdot & \cdot & \cdots & \cdot \\ y_{n+1}, & y_{n+2}, & \cdots & y_{2n+1} \end{vmatrix} = 0,$$

where $y_r = (D_x{}^r y)/r!$.

SOLUTION BY J. A. BULLARD, U. S. Naval Academy.

If $A = a_0 + a_1x + \cdots + a_nx^n$ and $B = b_0 + b_1x + \cdots + b_nx^n$, we have $By = A$ and, by Leibniz's Formula,

$$\sum_{h=0}^{h=n} \frac{(n+i)!}{h!\,(n+i-h)!} (D_x{}^{n+i-h}y)(D_x{}^hB) = 0 \quad (i = 1, 2, \cdots, n+1),$$

since $D_x{}^hA = D_x{}^hB = 0$ for $k > n$. Dividing by $(n+i)!$ we may write using notation of problem

$$\sum_{h=0}^{h=n} y_{n+i-h}B_h = 0 \quad (i = 1, 2, \cdots, n+1).$$

Eliminating B_h ($h = 0, 1, 2, \cdots, n$) from these equations, we obtain

$$\begin{vmatrix} y_1, & y_2, & \cdots & y_{n+1} \\ y_2, & y_3, & \cdots & y_{n+2} \\ \cdot & \cdot & \cdots & \cdot \\ y_{n+1}, & y_{n+2}, & \cdots & y_{2n+1} \end{vmatrix} = 0.$$

3096 [1924, 402]. Proposed by W. J. SIDIS, New York City.

In a scale of numeration whose radix is prime,

(1) There cannot be four distinct digits whose cubes all end in the same digit.

(2) If the cubes of two distinct digits end alike (that is, in the same digit), there will always be a third such digit.

(3) Under the conditions of (2), given any digit except 0, there will be two digits whose cubes have the same last figure as the cube of the given digit.

An extension of (2) and (3) is possible. If n is an odd prime, then, if the nth powers of two distinct digits end alike, there will be a group of n distinct digits whose nth powers end alike (that is, in the same figure), and such a group may be made to include any given digit not 0.

As an extension of (1), we may say that, under the original conditions, there cannot be $n + 1$ digits whose nth powers end alike.

SOLUTION BY H. L. OLSON, Michigan Agricultural College.

Let the radix be a prime p, and let r be a primitive root, mod. p. If $a \not\equiv 0 \pmod{p}$, there is an integer α such that $a \equiv r^\alpha \pmod{p}$; then a number $x \equiv r^\xi \pmod{p}$ satisfies the congruence $x^n \equiv a \pmod{p}$ if and only if $n\xi \equiv \alpha \pmod{p-1}$. Thus if n is a prime and not a factor of $p - 1$, this congruence has one and only one solution for each value of α. If n is a prime factor

of $p-1$, this congruence has n solutions or no solution according as α is or is not divisible by n. Hence, in a scale of numeration whose radix is a prime p:

If n is not a factor of $p-1$, there is one and only one digit whose nth power ends in any given digit not 0. If n is a factor of $p-1$, there are n digits or no digit whose nth powers end in a digit $\alpha \equiv r^{\alpha}$ (mod. p) according as α is or is not divisible by n. 0 is the final digit of the nth power of 0 and of no other digit. We have now proved the extensions of (1), (2) and (3) to any prime exponent n.

To extend the theorems to a composite exponent n, we need only to observe that if d is the greatest common divisor of n and $p-1$, the congruence $n\xi \equiv \alpha$ (mod. $p-1$) has d roots or no root according as α is or is not divisible by d. In other words, in a scale of numeration whose radix is a prime p:

There are d (and only d) digits or no digit whose nth powers end in a given digit $a \equiv r^{\alpha}$ (mod. p) according as α is or is not divisible by d, the greatest common divisor of n and $p-1$. As before, we see that 0 is the final digit of the nth power of 0 and of no other digit.

3098 [1924, 455]. Proposed by JEAN WINSTON, University of Cincinnati.

Find the involutes of the parabola $y = x^2$.

SOLUTION BY E. M. BERRY, Purdue University.

The involutes of a given curve are the orthogonal trajectories of the family of straight lines tangent to the given curve.

The differential equation of the family of straight lines tangent to the parabola $y = x^2$ is

$$p^2 - 4px + 4y = 0, \qquad p = \frac{dy}{dx}, \tag{1}$$

as is easily seen by differentiating and substituting in the point-slope form for the equation of a line.

Substituting $-1/p$ for p in (1) we get

$$4p^2y + 4xp + 1 = 0 \tag{2}$$

as the differential equation of the orthogonal trajectories of the family of straight lines represented by (1).

Solving equation (2) for x we get

$$x = -\frac{1}{4p} - py. \tag{3}$$

On differentiating with respect to y this becomes

$$\frac{1}{p} = \frac{1}{4p^2}\frac{dp}{dy} - y\frac{dp}{dy} - p$$

or

$$\frac{dy}{dp} + \frac{p}{1+p^2}y = \frac{1}{4p(p^2+1)},$$

a linear equation whose solution is

$$y = \frac{1}{\sqrt{p^2+1}}\left\{\int \frac{dp}{4p\sqrt{p^2+1}} + c\right\}. \tag{4}$$

Putting $p = \tan\varphi$ and integrating we have

$$y = \tfrac{1}{4}\cos\varphi[\log\tan(\varphi/2) + c]. \tag{5}$$

Substituting (5) in (3) we get

$$x = -\tfrac{1}{4}[\cot\varphi + \sin\varphi\log\tan(\varphi/2) + c\sin\varphi]. \tag{6}$$

Equations (5) and (6) are the equations of the involutes of the parabola $y = x^2$. If $c = 0$, we have $x = 0$ and $y = 0$ when $\varphi = 90°$, which is the particular one of these curves which passes through the vertex.

Also solved by THEODORE BENNETT and PAUL CAPRON.

3100 [1924, 455]. Proposed by H. E. TREFETHEN, Colby College.

Show that the segment between the axes tangent to the astroid at any point and the radius of the fixed circle to its point of contact with the generating circle bisect each other.

SOLUTION BY S. F. BIBB, University of North Dakota.

Let P be a point on the branch APB of the astroid in the first quadrant, and let the generating circle through P touch at Q the fixed circle of radius $OQ = a$ with its center at the origin O. The equations of the astroid are

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta,$$

where $\theta = \angle AOQ$. The equation of the tangent at P, MPS, is

$$y + x\tan\theta = a\sin\theta.$$

The intercepts of the tangent on the x and y axes are, respectively, $OM = a\cos\theta$ and $OS = a\sin\theta$. Therefore OM and OS are the projections of OQ on the axes, and it then follows that $OMQS$ is a rectangle. Hence OQ and MPS bisect each other at N, where NQ is a diameter of the generating circle.

NOTE BY THE EDITORS. All of the above results are easily obtained from the geometric definition of the astroid without the use of equations or differentiation.

Also solved by THEODORE BENNETT, PAUL CAPRON, and the PROPOSER.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

Among the grants in aid of research announced by the American Association for the Advancement of Science for 1925 are the following:

$250 to Dr. H. W. STAGER, of the University of Washington, for building a computing machine for use in constructing a list of prime numbers and a factor table; $300 to Professor A. O. LEUSCHNER, of the University of California, for use in the study of the perturbations of the minor planets.

On the occasion of the third Pan-American Scientific Congress, the University of San Marcos, at Lima, conferred the degree of Honorary Doctor in the Faculty of Sciences on Professor E. V. HUNTINGTON, of Harvard University. Professor Huntington attended the Congress as delegate of the American Mathematical Society, the American Academy of Arts and Sciences, and the Mathematical Association of America.

Assistant Professor V. B. HINSCH, of the Missouri School of Mines, has been promoted to an associate professorship of mathematics.

Assistant Professor MARSTON MORSE, of Cornell University, has been appointed associate professor of mathematics at Brown University.

Dr. H. L. OLSON, of the University of Michigan, has been appointed assistant professor of mathematics at Michigan Agricultural College.

Dr. H. M. STONE has been appointed to an instructorship in mathematics at Columbia University.

Mr. J. S. GOLD has been promoted to be assistant professor of mathematics at Bucknell University.

Assistant Professor C. A. GARABEDIAN, of Northwestern University, has been appointed to an associate professorship of mathematics in the University of Cincinnati.

Miss LAURA F. MCDONOUGH, of the University of Pennsylvania, has been appointed head of the department of mathematics at the Moravian College for Women, Bethlehem, Pa.

Associate Professor LAO G. SIMONS, of Hunter College, has published a memoir on the "Introduction of algebra into American schools in the eighteenth century"; it appears as Bulletin, 1924, No. 18, of the U. S. Bureau of Education and can be obtained from the Superintendent of Documents, Washington, D. C., for fifteen cents per copy. "It is the purpose of this study to show that algebra entered into the American education of the eighteenth century, and to show further that we must seek some other reason for its presence than a practical need for it." The final chapter contains a chronological list of American textbooks in algebra to 1820 and a fairly extensive bibliography.

Mr. E. C. RHODES, who served for several years as assistant to KARL PEARSON, comes to America this summer from the School of Economics, University of London, to give two courses of lectures at the Summer Session of Northwestern University. One course is entitled "An introduction to statistical method," the other "Advanced mathematics of statistics"; for the latter a semester of calculus is a prerequisite.

After 37 years of service, Professor ALEXANDER ZIWET of the University of Michigan will retire at the end of this semester. At a dinner held recently at the Michigan Union, he was given the following testimonial signed by the 32 members of the Mathematics Club of the University:

"We, the members of the Mathematics Club of the University of Michigan, desire to express to you our deep appreciation not only of your long and sincere interest in the welfare of the Club and of mathematical science at this University but of your high scholarly attainments in many branches of learning.

"We hold you, Sir, in profound respect, but more than that, in love and affection. Our signatures as inscribed below attest our enduring sentiments toward you."

MEETING OF THE SOUTHERN CALIFORNIA SECTION.

The first regular meeting of the Southern California Section of the Mathematical Association of America was held in Los Angeles, on Saturday, February 28, 1925, Professor E. R. Hedrick presiding.

There were forty-six present, including the following members of the Association: O. W. Albert, M. A. Basoco, H. Bateman, M. M. Beenken, G. E. Berry, W. N. Birchby, F. P. Brackett, J. R. Campbell, Mrs. T. Clark, M. Collier, Mae E. Conn, P. H. Daus, G. G. Entz, H. E. Glazier, F. C. Hall, E. R. Hedrick, H. C. Hicks, G. H. Hunt, G. James, Deca Lodwick, G. F. McEwen, W. E. Mason, F. R. Morris, W. A. Newlin, L. E. Reynolds, Boris Podolsky, W. P. Russell, G. E. F. Sherwood, M. Skarstedt, D. V. Steed, H. C. Van Buskirk, L. E. Wear, H. C. Willett, Clyde Wolfe, E. Worthington.

A constitution was adopted and the following officers elected: Professor HARRY BATEMAN, California Institute of Technology, *Chairman*; Professor H. C. WILLETT, University of Southern California, *Vice-Chairman*; Professor P. H. DAUS, University of California, Southern Branch, *Secretary-Treasurer*; Professor P. H. DAUS, Professor W. P. RUSSELL, Pomona College, and Mr. G. R. LIVINGSTON, San Diego Junior College, *Program Committee*.

Professor Bateman then took the chair. The following papers were presented. A short abstract appears below.

(1) "Contrast between the two methods of expressing symmetric functions in terms of the coefficients of the corresponding algebraic equations." Professor O. W. ALBERT, University of Redlands.

(2) "Note on a table of discounts." Professor P. H. DAUS, University of California, Southern Branch.

(3) "A solution of the quintic equation." Professor GLENN JAMES, University of California, Southern Branch.

(4) "Mathematics of fluid convection as the mechanism of heat conduction in large bodies of water." Professor G. F. MCEWEN, Scripps Institute, University of California.

(5) "The generalized Pellian equation." Professor CLYDE WOLFE, California Institute of Technology.

1. One method is based on the fundamental theorem for symmetric functions and expresses the function as a part of the product of elementary symmetric functions. The other is based on the value of

$$S_k \equiv \sum x_1^k,$$

which may be found by using Newton's identities or Waring's formula. The former method is useful when the degree of every root is small and the number of roots large, while the latter method is useful when the degree of any root is large and the number of roots is small.